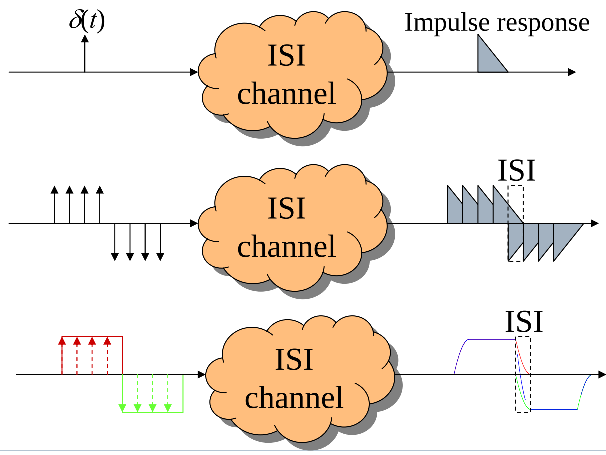
Chapter 4 Baseband Pulse Transmission

Techniques for the transmission of (originally) digital data over a baseband channel are the main focus of this chapter.

4.1 Introduction

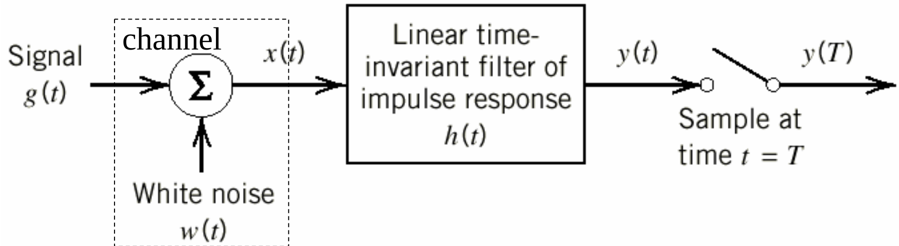
- Transmission of *digital data* (bit stream) over a *noisy* baseband channel typically suffers two channel imperfections
  - Intersymbol interference (ISI)
  - Background noise (e.g., AWGN)
- These two interferences/noises often occur simultaneously. However, for simplicity, they are often separately considered in analysis.

4.1 ISI



4.2 Matched filter

- Matched filter is a device for the optimal detection of a digital pulse. It is named so because the *impulse response* of the matched filter matches the *pulse shape*.
- System model without ISI



4.2 Design criterion

- To find $h(t)$ such that the output signal-to-noise ratio SNR_O is maximized.

$$x(t) = g(t) + w(t) \quad \text{for} \quad 0 \le t < T$$

$$\begin{aligned} y(t) &= [g(t) + w(t)] * h(t) \\ &= g(t) * h(t) + w(t) * h(t) \\ &= g_o(t) + n(t) \end{aligned}$$

$$SNR_O = \frac{|g_o(T)|^2}{E[n^2(T)]}$$

4.2 Analysis of matched filter

$$g_o(t) = \int_{-\infty}^{\infty} H(f)G(f)\exp(j2\pi ft)df$$

$$\Rightarrow |g_o(T)|^2 = \left|\int_{-\infty}^{\infty} H(f)G(f)\exp(j2\pi fT)df\right|^2$$

With $w(t)$ being white with PSD $N_0/2$,

$$S_N(f) = S_W(f)|H(f)|^2 = \frac{N_0}{2}|H(f)|^2$$

$$\Rightarrow E[n^2(T)] = \int_{-\infty}^{\infty} S_N(f)df = \frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2\, df$$

4.2 Analysis of matched filter

$$\Rightarrow \eta = \frac{\left|\int_{-\infty}^{\infty} G(f)H(f)\exp(j2\pi fT)df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\, df}$$

Cauchy-Schwarz inequality

$$\left|\int_{-\infty}^{\infty} \phi_1(x)\phi_2(x)dx\right|^2 \le \left(\int_{-\infty}^{\infty} |\phi_1(x)|^2\, dx\right)\left(\int_{-\infty}^{\infty} |\phi_2(x)|^2\, dx\right)$$

with equality holds if, and only if, $\phi_1(x) = k \cdot \phi_2^*(x)$ for some constant k.

4.2 Analysis of matched filter

- By Cauchy-Schwarz's inequality,

$$\left|\int_{-\infty}^{\infty} H(f)G(f)\exp(j2\pi fT)df\right|^2 \le \int_{-\infty}^{\infty} |H(f)|^2\, df \cdot \int_{-\infty}^{\infty} |G(f)\exp(j2\pi fT)|^2\, df$$

$$\Rightarrow \eta \le \frac{\int_{-\infty}^{\infty} |H(f)|^2\, df \cdot \int_{-\infty}^{\infty} |G(f)|^2\, df}{\frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\, df} = \frac{2}{N_0}\int_{-\infty}^{\infty} |G(f)|^2\, df$$

This is a constant bound, independent of the choice of $h(t)$.
Hence, the optimal η is achieved by:

$$\boxed{H(f) = k \cdot G^*(f)\exp(-j2\pi fT)}$$

4.2 Analysis of matched filter

$$h_{\text{opt}}(t) = \int_{-\infty}^{\infty} k \cdot G^*(f) \exp(-j2\pi fT) \exp(j2\pi ft) df$$
$$= k\left(\int_{-\infty}^{\infty} G(f) \exp(j2\pi f(T-t)) df\right)^*$$
$$= kg^*(T-t).$$

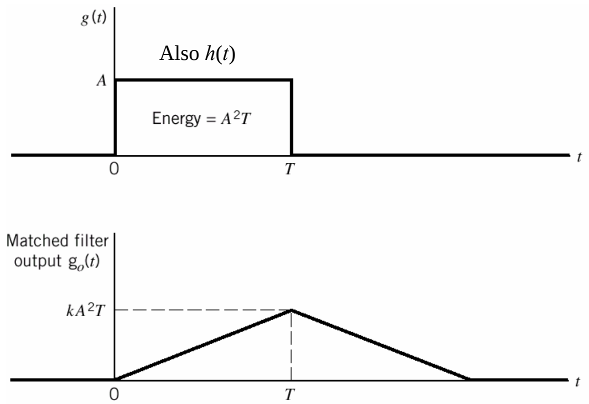
- Hence, under additive white noise, the *optimal received filter* matches the input signal in the sense that it is a time-inversed and delayed version of the complex-conjugated input signal $g(t)$.

4.2 Properties of matched filter

- The maximum output signal-to-noise ratio only depends on the energy of the input, and is nothing to do with the pulse shape itself.
  - Namely, whether the pulse shape is sinusoidal, rectangular, triangular, etc is irrelevant to the maximum output signal-to-noise ratio, as long as these pulse shapes have the same energy.

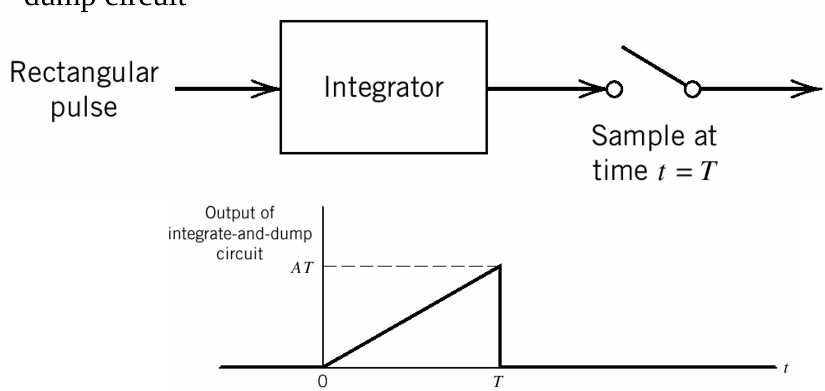
$$\eta_{\max} = \frac{2E_s}{N_0}, \text{ where } E_s = \int_{-\infty}^{\infty} |G(f)|^2 \, df.$$

Example 4.1 Matched filter for rectangular pulse



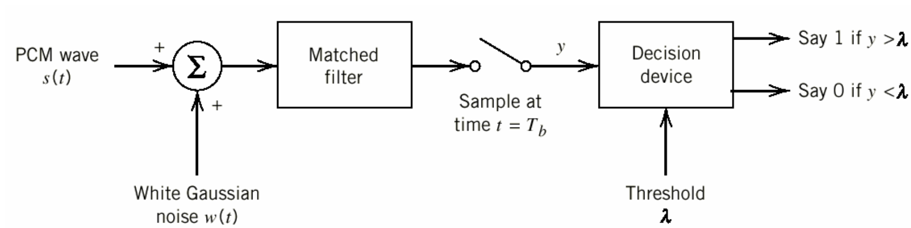
Example 4.1 Matched filter for rectangular pulse

- $h_{opt}(t)$ in this example can be implemented as integrate-and-dump circuit



4.3 Error rate due to noise

- In what follows, we analyze the error rate of *polar non-return-to-zero* (NRZ) *signaling* in a system with optimal matched filter receiver over AWGN channel.



$s(t) = I \cdot g(t)$, where $I \in \{-1, +1\}$.

$$
\begin{aligned}
y(T) &= [I \cdot g(t)] * h(t)\big|_{t=T} + w(t) * h(t)\big|_{t=T} \\
&= I \cdot \int_{-\infty}^{\infty} h(\tau) g(T-\tau) d\tau + \int_{-\infty}^{\infty} h(\tau) w(T-\tau) d\tau \\
&= I \cdot \int_{-\infty}^{\infty} k g^*(T-\tau) g(T-\tau) d\tau + \int_{-\infty}^{\infty} k g^*(T-\tau) w(T-\tau) d\tau \\
&= I \cdot k \int_{-\infty}^{\infty} |g(\tau)|^2 \, d\tau + k \int_{-\infty}^{\infty} g^*(\tau) w(\tau) d\tau \\
&= I \cdot k E_g + k n, \text{ where } E_g = \int_{-\infty}^{\infty} |g(\tau)|^2 \, d\tau \text{ and } n = \int_{-\infty}^{\infty} g^*(\tau) w(\tau) d\tau.
\end{aligned}
$$

For notational convenience, brief $y(T)/k$ by y.

(The integration can be taken over $[0, T)$ since $g(t)$ is zero outside this range, as does in text. I however use the entire real line as the integration range here for convenience.)

By AWGN assumption of $w(t)$, and real $g(t)$ assumption,

$n = \int_{-\infty}^{\infty} g^*(\tau) w(\tau) d\tau$ is Gussian distributed with

$$E[n] = \int_{-\infty}^{\infty} g^*(\tau) E[w(\tau)] d\tau = 0.$$

$$\begin{aligned} E[n^2] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(s) g(t) E[w(s)w(t)] ds dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(s) g(t) \frac{N_0}{2} \delta(s-t) ds dt \\ &= \frac{N_0}{2}\int_{-\infty}^{\infty} g^2(s) ds = \frac{N_0}{2} E_g \end{aligned}$$

$$y = I \cdot E_g + n \Rightarrow \begin{cases} \phi_{+1}(y) = Normal(E_g, E_g N_0/2), \text{if } I = +1; \\ \phi_{-1}(y) = Normal(-E_g, E_g N_0/2), \text{if } I = -1 \end{cases}$$

Let Ψ be the set for which a decision favors $+1$ is made.

$$\begin{aligned} BER &= \Pr[I=+1]\Pr\{\text{guess}\,(-1) \mid I=+1\} + \Pr[I=-1]\Pr\{\text{guess}\,(+1) \mid I=-1\} \\ &= \Pr[I=+1]\Pr\{y \notin \Psi \mid I=+1\} + \Pr[I=-1]\Pr\{y \in \Psi \mid I=-1\} \\ &= p\left(1-\Pr\{y\in\Psi \mid I=+1\}\right) + (1-p)\Pr\{y\in\Psi \mid I=-1\} \\ &= p + (1-p)\Pr\{y\in\Psi \mid I=-1\} - p\Pr\{y\in\Psi \mid I=+1\} \\ &= p + \int_{\Psi}[(1-p)\phi_{-1}(y) - p\phi_{+1}(y)]dy, \text{ where } p = \Pr[I=+1]. \end{aligned}$$

To minimize BER, the optimal set $\Psi_{\text{opt}} = \{y \in \Re : (1-p)\phi_{-1}(y) - p\phi_{+1}(y) < 0\}$.

Thus, the optimal decision maker is given by :

$$d(y) = \begin{cases} +1, & (1-p)\phi_{-1}(y) < p\phi_{+1}(y) \\ -1, & (1-p)\phi_{-1}(y) \ge p\phi_{+1}(y) \end{cases}.$$

$$\begin{cases}\phi_{+1}(y) = Normal(E_g, E_g N_0/2), \text{ if } I = +1;\\ \phi_{-1}(y) = Normal(-E_g, E_g N_0/2), \text{ if } I = -1\end{cases}$$

Let $\mu = E_g$ and $\sigma^2 = E_g N_0/2$.

$$\frac{(1-p)}{p} \underset{-1}{\overset{+1}{\lessgtr}} \frac{\phi_{+1}(y)}{\phi_{-1}(y)} = \frac{\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\}}{\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(y+\mu)^2}{2\sigma^2}\right\}}$$

$$= \exp\left\{\frac{2\mu y}{\sigma^2}\right\} = \exp\left\{\frac{2E_g y}{E_g N_0/2}\right\} = \exp\left\{\frac{4y}{N_0}\right\}$$

$$y \underset{-1}{\overset{+1}{\gtrless}} \frac{N_0}{4}\log\left[\frac{(1-p)}{p}\right]$$

This threshold depends on N_0; hence, the best decision relies on the accuracy of N_0 estimate.

4.3 Error rate due to noise under uniform input

- In order to free the system dependence on N_0 estimate, a uniform I is transmitted in which case, $p = ½$.
- The best decision now becomes $y \underset{-1}{\overset{+1}{\gtrless}} 0$.

$$\begin{aligned} BER_{\text{opt}} &= \frac{1}{2}\int_0^\infty \phi_{-1}(y)dy + \frac{1}{2}\int_{-\infty}^0 \phi_{+1}(y)dy \\ &= \frac{1}{2}\int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(y+\mu)^2}{2\sigma^2}\right\}dy \\ &\quad + \frac{1}{2}\int_{-\infty}^0 \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\}dy \end{aligned}$$

$$BER_{\text{opt}} = \frac{1}{2}\int_{\mu}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{y^2}{2\sigma^2}\right\} dy$$

$$+\frac{1}{2}\int_{-\infty}^{-\mu} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{y^2}{2\sigma^2}\right\} dy$$

$$= \int_{\mu}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{y^2}{2\sigma^2}\right\} dy, \; z = \frac{y}{\sqrt{2\sigma^2}}$$

$$= \frac{1}{\sqrt{\pi}} \int_{\mu/\sqrt{2\sigma^2}}^{\infty} \exp\left\{-z^2\right\} dz$$

$$= \frac{1}{2}\operatorname{erfc}\left(\frac{\mu}{\sqrt{2\sigma^2}}\right) = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_g}{N_0}}\right)$$

where $\operatorname{erfc}(u) = \frac{2}{\sqrt{\pi}}\int_u^{\infty} \exp(-z^2)dz$ is the complementary error function.

4.3 Error function

- Error function $\operatorname{erf}(u) = \frac{2}{\sqrt{\pi}}\int_0^{u} \exp(-z^2)dz$
- Complementary error function $\operatorname{erfc}(u) = \frac{2}{\sqrt{\pi}}\int_u^{\infty} \exp(-z^2)dz$
- Q-function $Q(u) = \frac{1}{\sqrt{2\pi}}\int_u^{\infty} \exp\left(-\frac{z^2}{2}\right)dz$

$$\begin{cases} \operatorname{erf}(-u) = -\operatorname{erf}(u) \\ \operatorname{erfc}(u) = 1 - \operatorname{erf}(u) \\ Q(u) = \frac{1}{2}\operatorname{erfc}\left(\frac{u}{\sqrt{2}}\right) \end{cases}$$

4.3 Error function

- Bounds for error function

$$\text{erfc}(x) = \frac{1}{x\sqrt{\pi}} e^{-x^2}\left(1 - \frac{1}{2x^2} + \frac{1\cdot 3}{2^2 x^4} - \frac{1\cdot 3\cdot 5}{2^3 x^6} + \cdots\right)$$

For $x > 0$, $\frac{1}{x\sqrt{\pi}} e^{-x^2}\left(1 - \frac{1}{2x^2}\right) < \text{erfc}(x) < \frac{1}{x\sqrt{\pi}} e^{-x^2}$

(The bound is good when x is large.)

4.3 Error rate due to noise

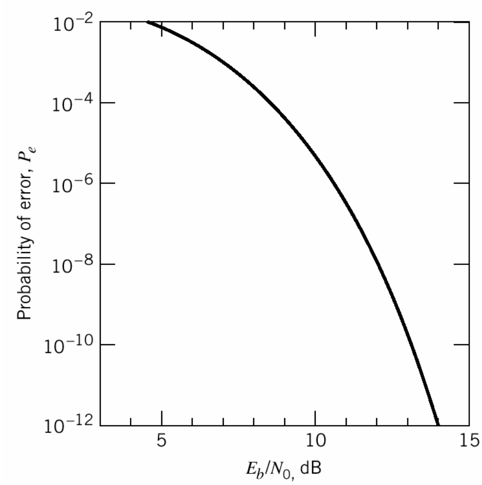
- The optimal BER formula is important in communications:

$$BER_{\text{opt}} = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{E_g}{N_0}}\right) = Q\left(\sqrt{\frac{2E_g}{N_0}}\right)$$

- The best decision is $y \underset{-1}{\overset{+1}{\gtrless}} 0$.

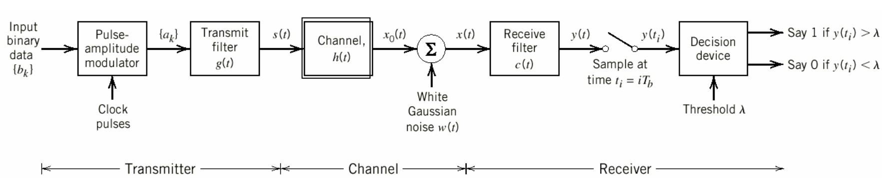
$s(t) = I \cdot g(t)$, where $I \in \{-1,+1\}$.

In this case, $E_b = \int_0^T E[s^2(t)]dt = \int_0^T E[I^2]g^2(t)dt = E_g$



4.4 Intersymbol interference

- The channel is usually *dispersive* in nature.
- In this section, we only consider discrete pulse-amplitude modulation (PAM). Consideration of PDM and PPM will be out of the scope of this section.



$$b_k \in \{0,1\},\ a_k = 2b_k - 1 \text{ and } s(t) = \sum_{k=-\infty}^{\infty} a_k g(t - kT_b).$$

4.4 Intersymbol interference

- Notably, in the previous section, we only consider one interval of input.

$$s(t) = I \cdot g(t)$$

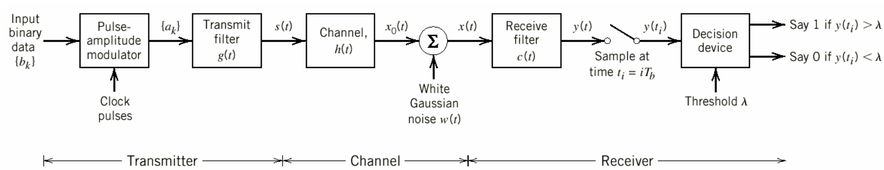
  This is justifiable because of no ISI.

- However, in this section, we have to consider

$$s(t) = \sum_{k=-\infty}^{\infty} a_k g(t - kT_b).$$

  since ISI is involved.

- We also assume *perfect synchronization* to simplify the analysis.



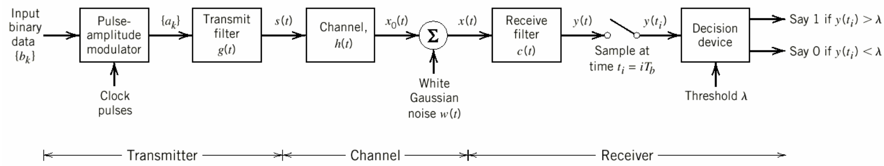
$$s(t) = \sum_{k=-\infty}^{\infty} a_k g(t - kT_b)$$

Information of a_k is carried at $[kT_b, (k+1)T_b)$.

$$x(t) = s(t) * h(t) + w(t) = \sum_{k=-\infty}^{\infty} a_k [g(t - kT_b) * h(t)] + w(t)$$

$$y(t) = x(t) * c(t) = \sum_{k=-\infty}^{\infty} a_k [g(t - kT_b) * h(t) * c(t)] + w(t) * c(t)$$

$$y(iT_b) = \sum_{k=-\infty}^{\infty} a_k [g(t - kT_b) * h(t) * c(t)] \Big|_{t=iT_b} + w(t) * c(t)\Big|_{t=iT_b}$$



$$g(t-kT_b)*h(t)*c(t) = \int_{-\infty}^{\infty} G(f)\exp\{-j2\pi fkT_b\}\cdot H(f)\cdot C(f)\exp\{j2\pi ft\}dt$$
$$= \int_{-\infty}^{\infty} G(f)\cdot H(f)\cdot C(f)\exp\{j2\pi f(t-kT_b)\}dt$$
$$= p(t-kT_b)$$

where $p(t) = \int_{-\infty}^{\infty} G(f)H(f)C(f)\exp\{j2\pi ft\}dt.$

$$\Rightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k p(t-kT_b) + n(t), \text{ where } n(t) = w(t)*c(t)$$

$$\Rightarrow y(iT_b) = \sum_{k=-\infty}^{\infty} a_k p(iT_b - kT_b) + n(iT_b)$$

4.4 ISI and noise

- Without ISI,

$$H(f) = 1 \Rightarrow p(t) = \int_{-\infty}^{\infty} G(f)C(f)\exp\{j2\pi ft\}dt.$$

With matched filter $C(f) = G^*(f)\exp\{-j2\pi fT_b\}$, or $c(t) = g^*(T_b - t)$,

$$p(iT_b) = \int_{-\infty}^{\infty} c(\tau)g(iT_b - \tau)d\tau$$
$$= \int_{-\infty}^{\infty} g^*(T_b - \tau)g(iT_b - \tau)d\tau, \;\; s = T_b - \tau$$
$$= \int_{-\infty}^{\infty} g^*(s)g(s + (i-1)T_b)ds$$
$$= 0, \text{ if } i \neq 1$$

$$y((i+1)T_b) = \sum_{k=-\infty}^{\infty} a_k p((i+1-k)T_b) + n((i+1)T_b) = a_i p(T_b) + n((i+1)T_b)$$

As a result, without ISI and additive noise,

$$y((i+1)T_b) = \sum_{k=-\infty}^{\infty} a_k p((i+1-k)T_b) = a_i p(T_b)$$

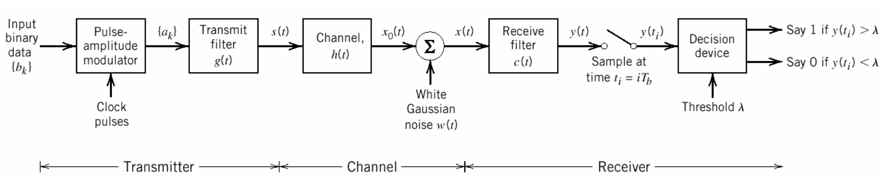
and $\{a_i\}$ can be completely reconstructed by $\{y((i+1)T_b)\}$.

The text sets $p(0) = 1$ for simplicity, but is a little confusion!
The text is correct when information of a_i is carried at $[(i-1)T_b, iT_b)$.

In my notations, information of a_i is actually carried at $[iT_b, (i+1)T_b)$.
So in order to recover a_i, "correlation" (convolution) operation should start at iT_b, and end (is sampled) at $(i+1)T_b$.
Hence, $y((i+1)T_b)$ is used to reconstruct a_i.

4.5 Nyquist's criterion for distortionless baseband binary transmission

- Is it possible to completely eliminate ISI (in principle) by selecting a proper $g(t)$?



Choose $g(t)$ and $c(t)$ such that $p(t) = \int_{-\infty}^{\infty} G(f)H(f)C(f)\exp\{j2\pi ft\}dt$

satisfies $p(iT_b) = 0$ for all $i \neq 0$, and $p(0) \neq 0$.

(Here, I assume that information of a_i is carried at $[(i-1)T_b, iT_b)$.)

4.5 Nyquist's criterion for distortionless baseband binary transmission

- Let $P(f) = G(f)H(f)C(f)$.
- Sample $p(t)$ with sampling period T_b to produce $P_\delta(f)$.
- (*Aliasing*) From slide Chapter 3-4, we get:

$$P_\delta(f) = \frac{1}{T_b}\sum_{n=-\infty}^{\infty} P\left(f - \frac{n}{T_b}\right)$$

- Also from slide Chapter 3-4, we have:

$$P_\delta(f) = \sum_{n=-\infty}^{\infty} p(nT_b)\exp(-j2\pi nT_b f) = 1$$

4.5 Nyquist's criterion for distortionless baseband binary transmission

- This concludes that the condition for zero ISI is:

$$\sum_{n=-\infty}^{\infty} P\left(f - \frac{n}{T_b}\right) = T_b$$

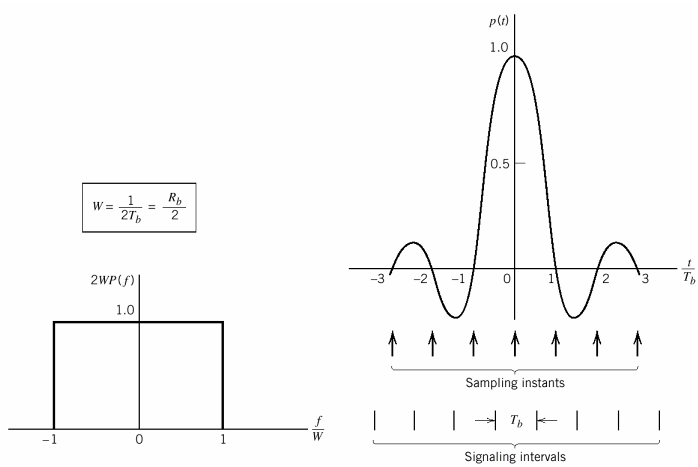
- This is named the *Nyquist criterion*.
  - The overall system frequency function $P(f)$ suffers no ISI for samples taken at interval T_b if it satisfies the above equation.
  - Notably, $P(f)$ represents the overall accumulative effect of transmit filter, channel response, receive filter, etc.

4.5 Ideal Nyquist channel

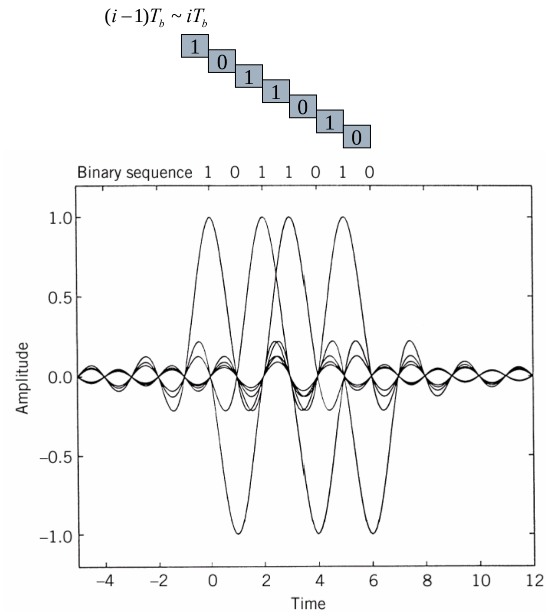
- The simplest $P(f)$ that satisfies Nyquist criterion is the rectangular function:

$$P(f)=\begin{cases} T_b, & |f|<W=\dfrac{1}{2T_b} \\ 0, & |f|>W=\dfrac{1}{2T_b} \end{cases} \text{ and } P(-W)+P(W)=T_b.$$

$$\Rightarrow p(t)=\frac{\sin(2\pi Wt)}{2\pi Wt}=\mathrm{sinc}(2Wt)$$



(a) (b)



4.5 Infeasibility of ideal Nyquist channel

- Rectangular $P(f)$ is infeasible because:
  - $p(t)$ extends to negative infinity, which means that each a_k have already been transmitted at $t = -\infty$!
  - A system response being flat from $-W$ to W, and zero elsewhere is physically unrealizable.
  - The margin of error is quite small, as a slight shift (error) in sampling time (such as, $iT_b+\varepsilon$) would cause very large ISI.
    - Note that $p(t)$ decays to zero at a very slow rate of $1/|t|$.

4.5 Infeasibility of ideal Nyquist channel

- Examination of timing error margin
  - Let Δt be the sampling time difference between transmitter and receiver.

$$y(iT_b + \Delta t) = \sum_{k=-\infty}^{\infty} a_k p((i-k)T_b + \Delta t)$$

  - For simplicity, set $i = 0$.

$$y(\Delta t) = \sum_{k=-\infty}^{\infty} a_k p(\Delta t - kT_b)$$

$$= \sum_{k=-\infty}^{\infty} a_k \frac{\sin[2\pi W(\Delta t - kT_b)]}{2\pi W(\Delta t - kT_b)}$$

$$y(\Delta t) = \sum_{k=-\infty}^{\infty} a_k \frac{\sin[2\pi W\Delta t - k\pi]}{2\pi W\Delta t - k\pi}$$

$$= \sum_{k=-\infty}^{\infty} a_k \frac{(-1)^k \sin[2\pi W\Delta t]}{2\pi W\Delta t - k\pi}$$

$$= a_0 \frac{\sin[2\pi W\Delta t]}{2\pi W\Delta t} + \frac{\sin[2\pi W\Delta t]}{\pi} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(-1)^k a_k}{2W\Delta t - k}$$

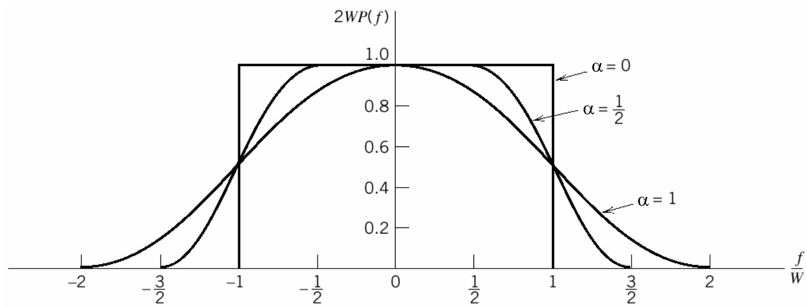
There exists $\{a_k\}$ such that $\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(-1)^k a_k}{2W\Delta t - k} = \infty$ for any fixed small $\Delta t > 0$.

Question: How to make $p(t)$ decays faster?
Answer: Make $P(f)$ smoother.

4.5 Raised Cosine Spectrum

For a nonnegative function $p(t)$,

$$\text{if } \int_{-\infty}^{\infty} t^k p(t)dt < \infty, \text{ then } \frac{\delta^k P(f)}{\delta f^k} \text{ exists.}$$



4.5 Raised Cosine Spectrum

- We extend the bandwidth of $p(t)$ from W to $2W$, and require that

$$P(f) + P(f-2W) + P(f+2W) = \frac{1}{2W} \quad \text{for } |f| < W.$$

  - So the price to pay here is a larger bandwidth.
  - One of the $P(f)$ that satisfies the above condition is the *raised cosine spectrum*.

$$P(f) = \begin{cases} \frac{1}{2W}, & 0 \le |f| < (1-\alpha)W \\ \frac{1}{4W}\left\{1+\cos\left[\frac{\pi(|f|-(1-\alpha)W)}{2\alpha W}\right]\right\}, & (1-\alpha)W \le |f| < (1+\alpha)W \\ 0, & |f| \ge (1+\alpha)W \end{cases}$$

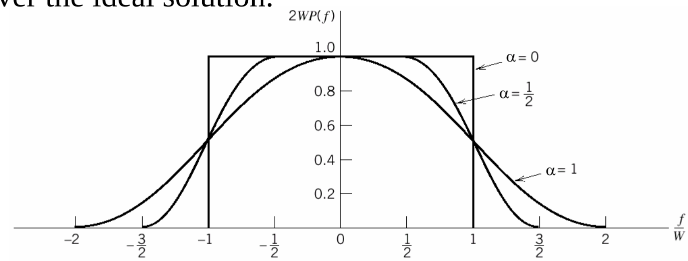
The text puts $B_T = W(1+\alpha)$ may not be justifiable!

4.5 Raised Cosine Spectrum

- The transmission bandwidth of the raised cosine spectrum is equal to:

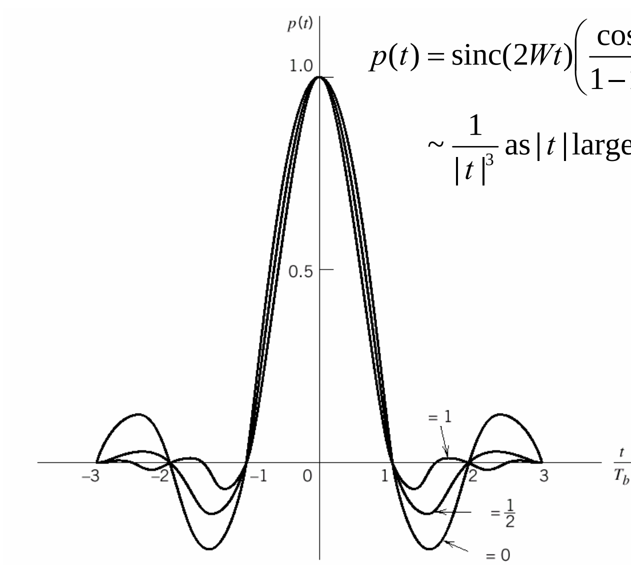
$$B_T = 2W(1+\alpha)$$

where α is the rolloff factor, which is the *excess bandwidth* over the ideal solution.



$$p(t) = \mathrm{sinc}(2Wt)\left(\frac{\cos(2\pi\alpha Wt)}{1-16\alpha^2W^2t^2}\right)$$

$$\sim \frac{1}{|t|^3} \text{ as } |t| \text{ large}$$



4.5 Raised Cosine Spectrum

- $p(t) = \mathrm{sinc}(2Wt)\left(\dfrac{\cos(2\pi\alpha Wt)}{1-16\alpha^2W^2t^2}\right)$ consists of two terms:
  - The first term ensures the desired zero crossing of $p(t)$.
  - The second term provides the necessary tail convergence rate of $p(t)$.
- The special case of $\alpha = 1$ is known as the *full-cosine rolloff* characteristic.

$$p(t) = \frac{\mathrm{sinc}(4Wt)}{1-16W^2t^2}$$

4.5 Raised Cosine Spectrum

- Useful property of *full-cosine spectrum*.

$$p\left(\pm\frac{iT_b}{2}\right) = \begin{cases} 1, & i = 0 \\ \dfrac{1}{2}, & i = 1 \\ 0, & i \geq 2 \end{cases}$$

  - We have more "zero-crossing" at $\pm 3T_b/2$, $\pm 5T_b/2$, $\pm 7T_b/2, \ldots$ in addition to the desired $\pm T_b$, $\pm 2T_b$, $\pm 3T_b \ldots$
  - This is useful in synchronization. (Think of when "synchronized", the quantity should be small both at $\pm 3T_b/2$, $\pm 5T_b/2$, $\pm 7T_b/2, \ldots$ and at $\pm T_b$, $\pm 2T_b$, $\pm 3T_b \ldots$)
  - However, the price to pay for this excessive synchronization information is to "double the bandwidth".

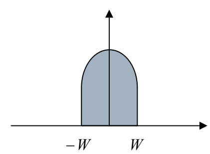
Example 4.2 Bandwidth requirement of the T1 system

- For T1 transmission, a frame consists of 24 PCM-encoded voice channels and 1 framing bit.
  - The resultant number of bits in a frame is 24 × 8 + 1 = 193.
- The duration of each frame is 125μs.
- Hence,

$$T_b = \frac{125\mu s}{193} = 0.647688\mu s$$

$$\Rightarrow W = \frac{1}{2T_b} = 772\text{kHz}.$$

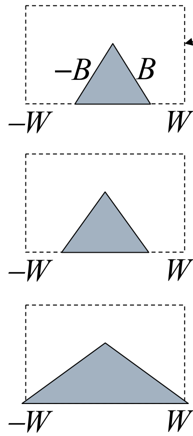
$$\Rightarrow B_{T,(\text{Baseband})} = W(1+\alpha) = 772(1+\alpha)\text{kHz}$$



4.6 Correlative-level coding

- ISI, when generated in an uncontrolled manner, is an undesirable phenomenon.
- However, ISI may become a friend if it is added to the transmitted signal in a controlled manner.
  - *Known fact*: A signal of bandwidth W can be distortionlessly transmitted using its samples with sampling rate ≥ $2W$.
  - Conversely, in a channel with bandwidth WHz, the theoretical maximum signal rate is $2W$ symbols per second.

4.6 Correlative-level coding



A channel with bandwidth W Hz

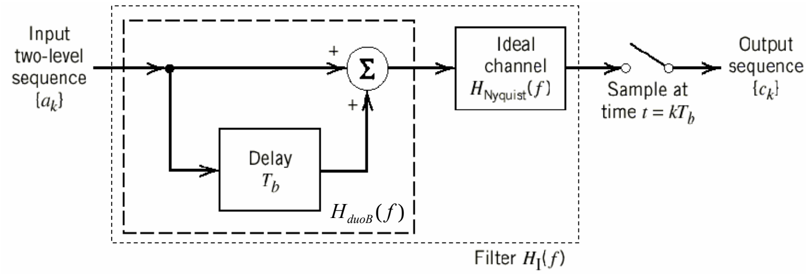
The maximum signal rate is $2W$ samples per second.

4.6 Correlative-level coding

- Why intentionally adding ISI? Answer: To have better bandwidth efficiency.
  - **Ideal Nyquist pulse shaping** is efficient; it cannot be realized.
  - **Raised consine pulse shaping** is realizable; it is bandwidth inefficient.
  - By *adding ISI* to the transmitted symbols in a controlled manner, we can achieve the Nyqusit rate $2W$ in a channel bandwidth of W Hertz.
    - Correlative-level coding or Partial-response signaling

4.6 One example of Correlative-level coding

- Duobinary signaling (or *class I partial response*)



$$a_k = \begin{cases} +1 & \text{if symbol } b_k \text{ is } 1 \\ -1 & \text{if symbol } b_k \text{ is } 0 \end{cases} \quad \text{where } \{b_k\} \text{ i.i.d.}$$

4.6 Duobinary signaling

- Let's ignore the effect of $H_{\text{Nyquist}}(f)$ first in the block diagram in the previous slide. We directly obtain:

$$c_k = a_k + a_{k-1}$$

$$\Rightarrow H_{DuoB}(f) = 1 + \exp(-j2\pi f T_b)$$

  - Note that c_k has three levels (–2,0,2).
- The transfer function of the overall system is thus:

$$\begin{aligned} H_{\text{I}}(f) &= H_{\text{Nyquist}}(f)[1 + \exp(-j2\pi f T_b)] \\ &= H_{\text{Nyquist}}(f)[\exp(j\pi f T_b) + \exp(-j\pi f T_b)] \exp(-j\pi f T_b) \\ &= 2H_{\text{Nyquist}}(f) \cos(\pi f T_b) \exp(-j\pi f T_b) \end{aligned}$$

4.6 Duobinary signaling

- $H_{\text{Nyqusit}}(f)$:
  - Only for derivation purpose (do not need it finally)

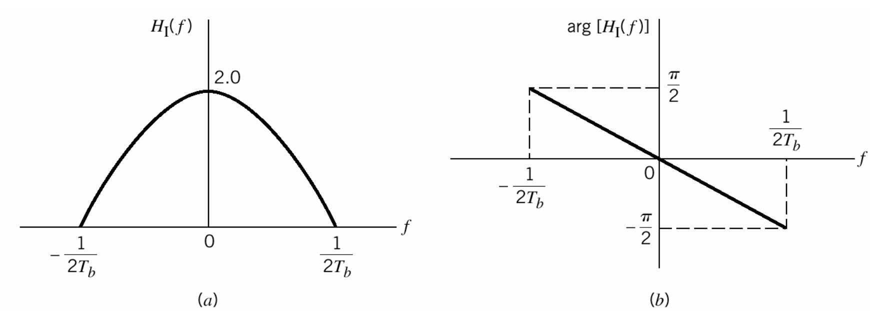
$$H_{\text{Nyquist}}(f) = \begin{cases} 1, & |f| \le 1/2T_b \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow H_{\text{I}}(f) = \begin{cases} 2\cos(\pi f T_b)\exp(-j\pi f T_b), & |f| \le 1/2T_b \\ 0, & \text{otherwise} \end{cases}$$

  - As shown in the next slide, the response $H_{\text{I}}(f)$ is realizable.

4.6 Duobinary signaling

- $H_I(f)$



4.6 Duobinary signaling

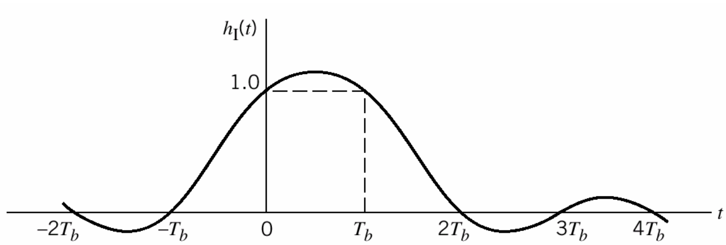
- $h_I(t)$:

$$H_{\mathrm{I}}(f) = H_{\mathrm{Nyquist}}(f)[1 + \exp(-j2\pi f T_b)]$$
$$= H_{\mathrm{Nyquist}}(f) + H_{\mathrm{Nyquist}}(f)\exp(-j2\pi f T_b)]$$

$$\Rightarrow h_{\mathrm{I}}(t) = h_{\mathrm{Nyquist}}(t) + h_{\mathrm{Nyquist}}(t - T_b)$$
$$= \frac{\sin(\pi t/T_b)}{\pi t/T_b} + \frac{\sin[\pi(t - T_b)/T_b]}{\pi(t - T_b)/T_b}$$
$$= \frac{\sin(\pi t/T_b)}{\pi t/T_b} - \frac{\sin(\pi t/T_b)}{\pi(t - T_b)/T_b}$$
$$= \frac{T_b^2 \sin(\pi t/T_b)}{\pi t(T_b - t)}$$

4.6 Duobinary signaling

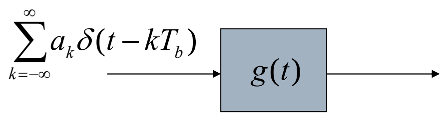
- $h_I(t)$:



4.6 Duobinary signaling

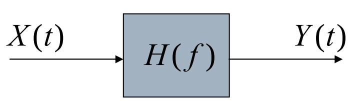
- **Bandwidth efficiency** of duobinary signaling
  - Example.

$$\text{The transmitted signal} = \sum_{k=-\infty}^{\infty} a_k g(t-kT_b) = \left[\sum_{k=-\infty}^{\infty} a_k \delta(t-kT_b)\right] * g(t)$$



The input to this filter is not WSS!
Then we can introduce time-average autocorrelation function.

4.6 Time-average autocorrelation function



Time Average Autocorrelation Function

$$\overline{R}_X(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[X(t)X(t+\tau)]dt = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_X(t,t+\tau)dt$$

$$\Rightarrow \quad \overline{R}_Y(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_Y(t,t+\tau)dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(t-\tau_1,t+\tau-\tau_2)d\tau_1 d\tau_2 dt$$

(Assume that *limit* and *integration* are interchangeable.)

$$\overline{R}_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_X(t-\tau_1, t+\tau-\tau_2)dt\right)d\tau_1 d\tau_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)\overline{R}_X(\tau-\tau_2+\tau_1)d\tau_1 d\tau_2$$

$$\overline{S}_Y(f) = \int_{-\infty}^{\infty} \overline{R}_Y(\tau)\exp(-j2\pi f\tau)d\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)\int_{-\infty}^{\infty} \overline{R}_X(\tau-\tau_2+\tau_1)\exp(-j2\pi f\tau)d\tau d\tau_1 d\tau_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)\int_{-\infty}^{\infty} \overline{R}_X(\tau')\exp(-j2\pi f[\tau'+\tau_2-\tau_1])d\tau' d\tau_1 d\tau_2$$

$$= \int_{-\infty}^{\infty} h(\tau_2)e^{-j2\pi f\tau_2}d\tau_2 \cdot \int_{-\infty}^{\infty} h(\tau_1)e^{j2\pi f\tau_1}d\tau_1 \cdot \int_{-\infty}^{\infty} \overline{R}_X(\tau')e^{-j2\pi f\tau'}d\tau'$$

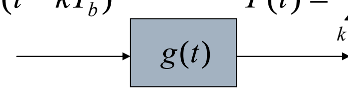
$$= H(f)H^*(f)\overline{S}_X(f), \text{ if } h(\tau) \text{ is real}$$

$$= |H(f)|^2\ \overline{S}_X(f)$$

4.6 Duobinary signaling

- Now back to the example.

$$X(t) = \sum_{k=-\infty}^{\infty} a_k\delta(t-kT_b) \qquad Y(t) = \sum_{k=-\infty}^{\infty} a_k g(t-kT_b)$$



(to channel)

$$\overline{R}_X(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E\left[\left(\sum_{k=-\infty}^{\infty} a_k\delta(t-kT_b)\right)\left(\sum_{j=-\infty}^{\infty} a_j\delta(t+\tau-jT_b)\right)\right]dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} E[a_j a_k]\delta(t+\tau-jT_b)\delta(t-kT_b)dt$$

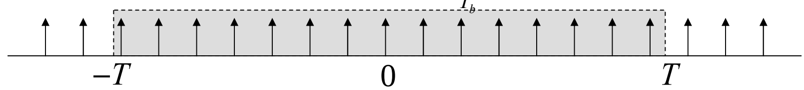
4.6 Duobinary signaling

$$\overline{R}_X(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(\sum_{k=-\infty}^{\infty} \delta(t - kT_b)\delta(t + \tau - kT_b) \right) dt$$

$$= \delta(\tau) \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(\sum_{k=-\infty}^{\infty} \delta(t - kT_b) \right) dt$$

$$= \frac{1}{T_b}\delta(\tau) \quad \Rightarrow \overline{S}_Y(f) = \frac{1}{T_b} |G(f)|^2$$

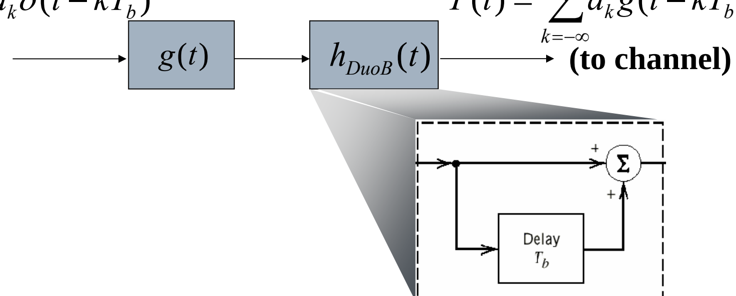
Approximately $\frac{2T}{T_b}$ of them



4.6 Duobinary signaling

$$X(t) = \sum_{k=-\infty}^{\infty} a_k \delta(t - kT_b)$$

$$Y(t) = \sum_{k=-\infty}^{\infty} a_k g(t - kT_b)$$



$$\Rightarrow \overline{S}_Y(f) = \frac{1}{T_b} |G(f)|^2 |H_{DuoB}(f)|^2$$

4.6 Duobinary signaling

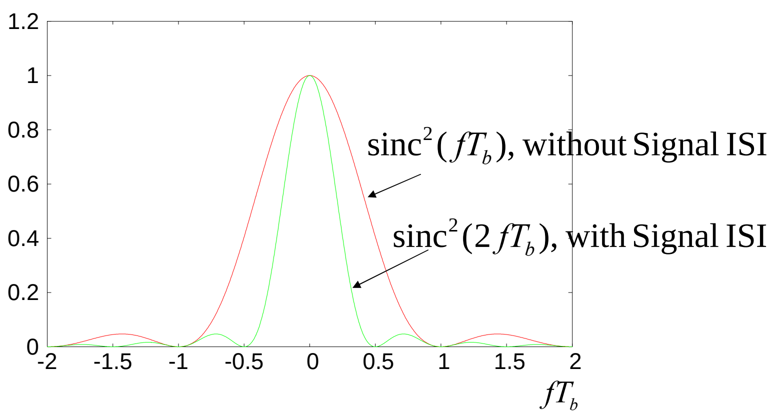
$$H_{DouB}(f) = 2\cos(\pi f T_b)\exp(-j\pi f T_b)$$

$$\text{Assume } g(t) = \begin{cases} 1, & 0 \le t < T_b \\ 0, & \text{otherwise} \end{cases}. \Rightarrow |G(f)|^2 = T_b^2 \operatorname{sinc}^2(fT_b)$$

$$\Rightarrow \frac{\bar{S}_Y(f)}{\bar{S}_Y(0)} = \begin{cases} \operatorname{sinc}^2(fT_b), & \text{No Signal ISI} \\ \cos^2(\pi f T_b)\operatorname{sinc}^2(fT_b), & \text{With Signal ISI} \end{cases}$$

$$= \begin{cases} \operatorname{sinc}^2(fT_b), & \text{No Signal ISI} \\ \operatorname{sinc}^2(2fT_b), & \text{With Signal ISI} \end{cases}$$

4.6 Duobinary signaling



4.6 Duobinary signaling

- Conclusions
  - By adding ISI to the transmitted signal in a controlled (and reversible) manner, we can reduce the requirement of bandwidth of the transmitted signal.
  - Hence, in the previous example, $\{c_k\}$ can be transmitted in every $T_b/2$ seconds!
    - Doubling the transmission capacity without introducing additional requirement in bandwidth!
  - *Duobinary signaling* : “Duo” means “doubling the transmission capacity of a straight binary system.
  - A large SNR is required to yield the same error rate because of an increase in the number of signal levels (from –1, +1 to –2, 0, 2). Detailed discussion on error rate impact is omitted here!

4.6 Duobinary signaling

- Conclusions (cont.)
  - The duobinary signaling is also named *class I partial response*.
    - Full response: The transmission wave at each time instance is fully determined by a single information symbol.
    - Partial response: The transmission wave at each time instance is only partially determined by one information symbol (i.e., is fully determined by two or more information symbols).

4.6 Decision feedback for correlative-level coding

- Recovering of $\{a_k\}$ from $\{c_k\}$

$$\hat{a}_k = c_k - \hat{a}_{k-1}$$

  - It requires the previous decision to determine the current symbol.
  - So the system should feedback the previous decision.
  - Error therefore may propagate!
  - How to avoid error propagation? Answer: **Precoding**.

4.6 Precoding of correlative coding

Without precoding

$$\boxed{\{b_k \in \{0,1\} \text{ i.i.d.}\}} \to \boxed{a_k = 2b_k - 1} \to \boxed{c_k = a_k + a_{k-1}}$$

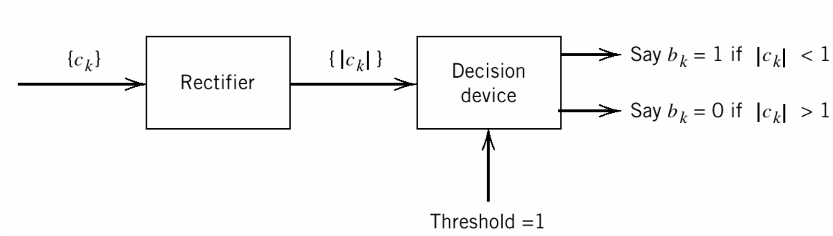
With precoding

$$\boxed{\{b_k \in \{0,1\} \text{ i.i.d.}\}} \to \boxed{\tilde{b}_k = b_k \oplus \tilde{b}_{k-1}} \to \boxed{a_k = 2\tilde{b}_k - 1} \to \boxed{c_k = a_k + a_{k-1}}$$

$$\begin{cases} c_k = a_k + a_{k-1} \\ \quad = (2\tilde{b}_k - 1) + (2\tilde{b}_{k-1} - 1) \\ \quad = 2\tilde{b}_k + 2\tilde{b}_{k-1} - 2 \\ b_k = \tilde{b}_k \oplus \tilde{b}_{k-1} \end{cases}$$

$\tilde{b}_k$	$\tilde{b}_{k-1}$	b_k	c_k
0	0	0	−2
0	1	1	0
1	0	1	0
1	1	0	2

4.6 Precoding of correlative coding



- Final notes
  - The precode must not change the "**duo-** of the transmission capacity of a straight binary system."
  - Hence, $\{\tilde{b}_k\}$ must be i.i.d.

4.6 Precoding of correlative coding

- I.i.d. of $\{\tilde{b}_k\}$
  - It suffices to show that: $\Pr\left(\tilde{b}_k \mid \tilde{b}_{k-1}, \tilde{b}_{k-2}, \ldots\right) = \Pr\left(\tilde{b}_k\right)$

$$\tilde{b}_k = b_k \oplus \tilde{b}_{k-1} \Rightarrow \Pr\left(\tilde{b}_k \mid \tilde{b}_{k-1}, \tilde{b}_{k-2}, \ldots\right) = \Pr\left(\tilde{b}_k \mid \tilde{b}_{k-1}\right)$$

$$\begin{cases} \Pr(\tilde{b}_k = 0 \mid \tilde{b}_{k-1} = 0) = \Pr(b_k = 0) = 1/2 \\ \Pr(\tilde{b}_k = 0 \mid \tilde{b}_{k-1} = 1) = \Pr(b_k = 1) = 1/2 \\ \Pr(\tilde{b}_k = 1 \mid \tilde{b}_{k-1} = 0) = \Pr(b_k = 1) = 1/2 \\ \Pr(\tilde{b}_k = 1 \mid \tilde{b}_{k-1} = 1) = \Pr(b_k = 0) = 1/2 \end{cases}$$

$$
\begin{cases}
\Pr(\tilde{b}_k = 0) = \Pr(\tilde{b}_{k-1} = 0)\Pr(\tilde{b}_k = 0 \mid \tilde{b}_{k-1} = 0) \\
\qquad\qquad + \Pr(\tilde{b}_{k-1} = 1)\Pr(\tilde{b}_k = 0 \mid \tilde{b}_{k-1} = 1) \\
\qquad = \Pr(\tilde{b}_{k-1} = 0)\dfrac{1}{2} + \Pr(\tilde{b}_{k-1} = 1)\dfrac{1}{2} \\
\qquad = \dfrac{1}{2} \\
\Pr(\tilde{b}_k = 1) = \Pr(\tilde{b}_{k-1} = 0)\Pr(\tilde{b}_k = 1 \mid \tilde{b}_{k-1} = 0) \\
\qquad\qquad + \Pr(\tilde{b}_{k-1} = 1)\Pr(\tilde{b}_k = 1 \mid \tilde{b}_{k-1} = 1) \\
\qquad = \Pr(\tilde{b}_{k-1} = 0)\dfrac{1}{2} + \Pr(\tilde{b}_{k-1} = 1)\dfrac{1}{2} \\
\qquad = \dfrac{1}{2}
\end{cases}
$$

Q.E.D.

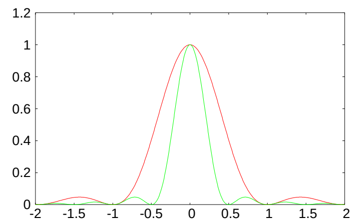
Example 4.3 Duobinary coding with precoding

- Table 4.1 in text

$\{b_k\}$		0	0	1	0	1	1	0
$\{\tilde{b}_k\}$	1	1	1	0	0	1	0	0
$\{a_k\}$	+1	+1	+1	–1	–1	+1	–1	–1
$\{c_k\}$		+2	+2	0	–2	0	0	–2
$\{\hat{b}_k\}$		0	0	1	0	1	1	0

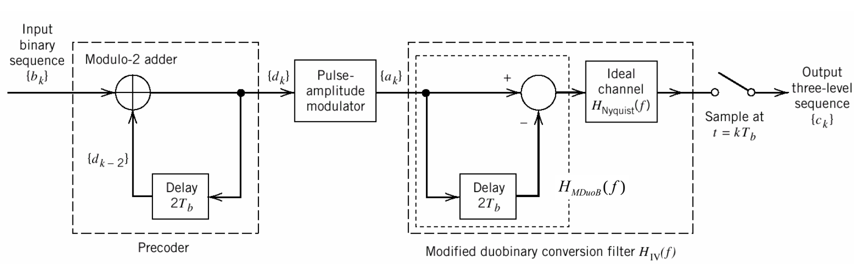
4.6 Modified duobinary signaling

- The PSD of the signal is nonzero at the origin.
- The is considered to be an **undesirable feature** in some applications, since many communication channels cannot transmit a DC component.
- Solution: Class IV partial response or modified duobinary technique.



4.6 Modified duobinary signaling

$$\{b_k \in \{0,1\} \text{ i.i.d.}\} \to \boxed{\tilde{b}_k = b_k \oplus \tilde{b}_{k-2}} \to \boxed{a_k = 2\tilde{b}_k - 1} \to \boxed{c_k = a_k - a_{k-2}}$$



$$\Rightarrow H_{MDuoB}(f) = 1 - \exp(-j4\pi f T_b)$$

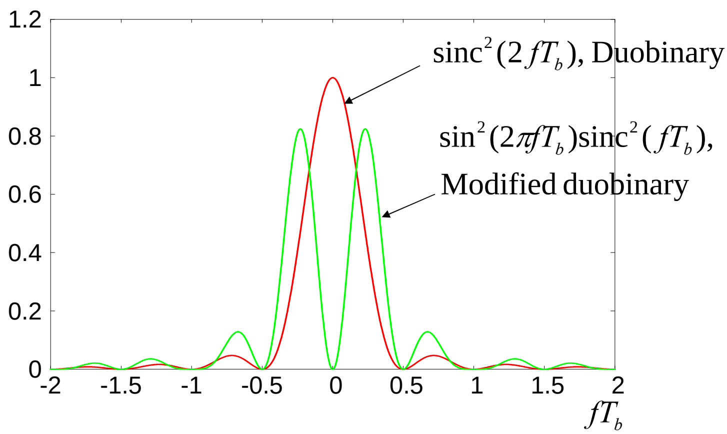
4.6 Modified duobinary signaling

$$\Rightarrow H_{MDuoB}(f) = 1 - \exp(-j4\pi f T_b)$$
$$= [\exp(j2\pi f T_b) - \exp(-j2\pi f T_b)]\exp(-j2\pi f T_b)$$
$$= 2j\sin(2\pi f T_b)\exp(-j2\pi f T_b)$$

Assume $g(t) = \begin{cases} 1, & 0 \le t < T_b \\ 0, & \text{otherwise} \end{cases}$. $\Rightarrow |G(f)|^2 = T_b^2 \operatorname{sinc}^2(fT_b)$

$$\Rightarrow \begin{cases} \bar{S}_Y(f)/T_b = \operatorname{sinc}^2(2fT_b), & \text{Duobinary} \\ \bar{S}_Y(f)/(4T_b) = \sin^2(2\pi f T_b)\operatorname{sinc}^2(fT_b), & \text{Modified Duobinary} \end{cases}$$

4.6 Modified duobinary signaling



4.6 Modified duobinary signaling

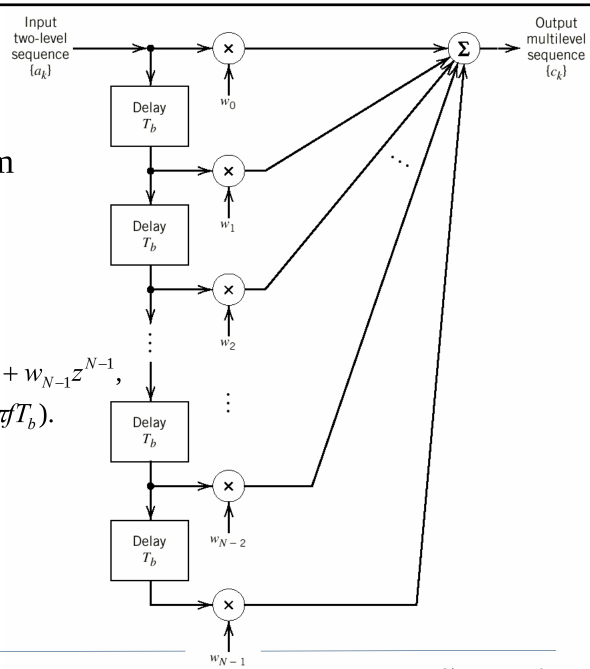
- Precoding is added to eliminate *error propagation* in decision system.

$$
\begin{cases}
c_k = a_k - a_{k-2} \\
\quad = (2\tilde{b}_k - 1) - (2\tilde{b}_{k-2} - 1) \\
\quad = 2\tilde{b}_k - 2\tilde{b}_{k-2} \\
b_k = \tilde{b}_k \oplus \tilde{b}_{k-2}
\end{cases}
$$

$\tilde{b}_k$	$\tilde{b}_{k-2}$	b_k	c_k
0	0	0	0
0	1	1	–2
1	0	1	2
1	1	0	0

4.6 Generalized form of correlative level coding (or partial response signaling)

$$H_{CLC}(f) = w_0 + w_1 z^{-1} + \cdots + w_{N-1} z^{N-1},$$
$$\text{where } z = \exp(-j2\pi f T_b).$$

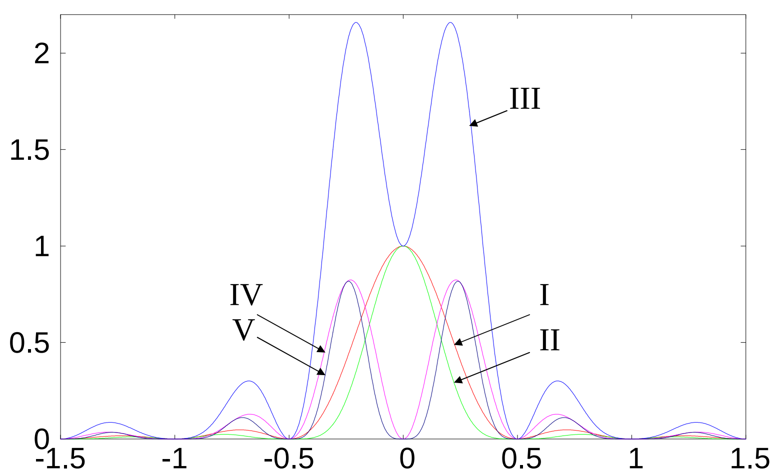


4.6 Generalized form of correlative-level coding (or partial-response signaling)

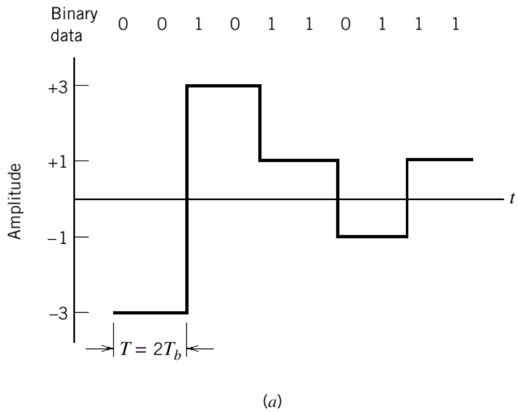
Type of Class	N	w_0	w_1	w_2	w_3	w_4	Comments
I	2	1	1				Duobinary coding
II	3	1	2	1			
III	3	2	1	−1			
IV	3	1	0	−1			Modified duobinary coding
V	5	−1	0	2	0	−1	

$$\Rightarrow \bar{S}_Y(f) = \frac{|G(f)|^2}{T_b} \times \begin{cases} 4\cos^2(\pi f T_b) & I \\ 16\cos^4(\pi f T_b) & II \\ 4\cos^2(\pi f T_b) + 8\sin^2(2\pi f T_b) & III \\ 4\sin^2(2\pi f T_b) & IV \\ 16\sin^4(2\pi f T_b) & V \end{cases}$$

Assume $g(t) = \begin{cases} 1, & 0 \le t < T_b \\ 0, & \text{otherwise} \end{cases}$. $\Rightarrow |G(f)|^2 = T_b^2 \operatorname{sinc}^2(fT_b)$



4.7 Baseband M-ary PAM transmission



Dibit	Amplitude
00	−3
01	−1
11	+1
10	+3

Gray code

Any dibit differs from an adjacent dibit in a single bit position.

4.7 Baseband M-ary PAM transmission

- For M-ary PAM transmission, there are M possible symbols with symbol duration T.
  - $1/T$ is referred to as the *signaling rate* or *symbol rate* or *symbols per second* or *baud*.
- Some equivalences
  - Each symbol can be equivalently identified with $\log_2 M$ bits.
  - So the baud rate $1/T$ can be equivalently transformed to bps as:

$$T = T_b \log_2(M)$$

4.7 Baseband *M*-ary PAM transmission

- Some equivalences
  - Virtually fix the symbol error, namely, fix the level distance (to be 2). For example, (+1, −1) for M = 2, and (+3, +1, −1, −3) for M = 4. Then the transmitted power per unit time for *M*-ary PAM transmission becomes:

$$\frac{E[S^2]}{T} = \frac{\frac{1}{M}\left([-(M-1)]^2 + [-(M-3)]^2 + \cdots + (M-3)^2 + (M-1)^2\right)}{T_b \log_2(M)}$$

$$= \frac{(M^2-1)}{3T_b \log_2(M)} = \left(\frac{1}{T_b}\right)\frac{(M^2-1)}{3\log_2(M)}$$

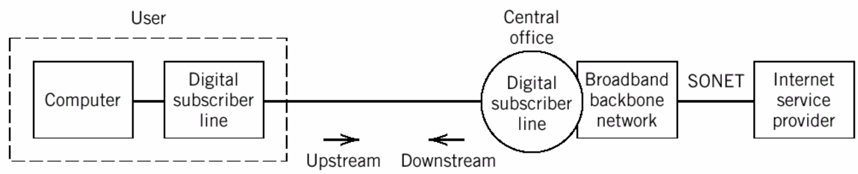
4.7 Baseband *M*-ary PAM transmission

$$\frac{E[S^2]}{T} = \left(\frac{1}{T_b}\right)\frac{(M^2-1)}{3\log_2(M)}$$

For fixed "bps", the transmitted power for *M*-ary transmission must be increased by a factor $M^2/\log_2 M$.

4.8 Digital subscriber lines

- A DSL operates over a local loop (often less than 1.5km) that provides a direct connection between a user terminal (e.g., computer) and a telephone company's *central office* (CO).
  - Since it is a direct connection, no dialup is necessary.
  - The information-bearing signal is kept in the digital domain all the way from the user terminal to an Internet service provider.

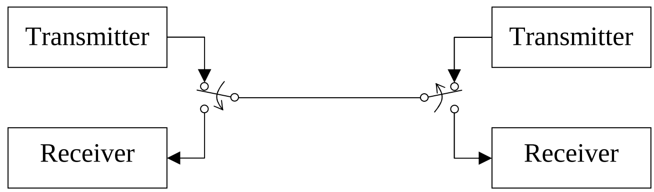


4.8 Digital subscriber lines

- DSL is intended to provide *high data-rate*, *full-duplex*, *digital* transmission capability using local cost configuration (such as twisted pairs for ordinary telephonic communications).
- One of two possible modes can be used to achieve the full-duplex goal.
  - Time compression (TC) multiplexing
  - Echo-cancellation mode

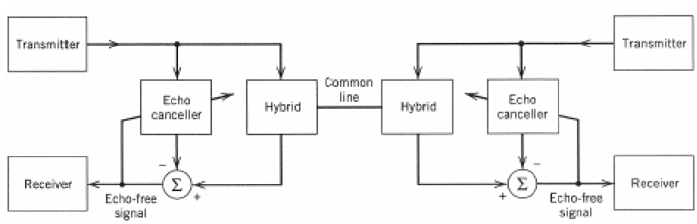
4.8 Digital subscriber lines

- Time-compression (TC) multiplexing
  - A guard time is often inserted between bursts in the two opposite directions of data.
  - So the required line rate is slightly greater than twice the data rate.



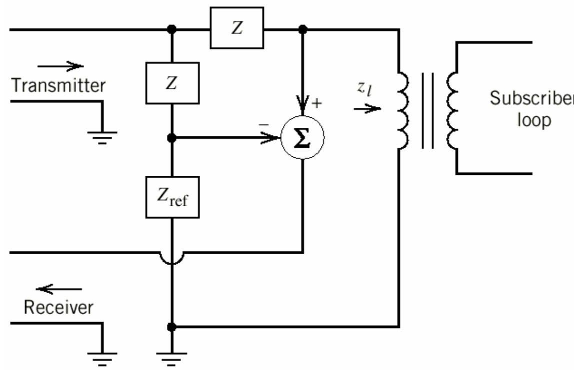
4.8 Digital subscriber lines

- Echo-cancellation (EC) mode
  - Support the simultaneous flow of data along the common line in both directions.
  - In this mode, the line rate is the same as the data rate.



4.8 Digital subscriber lines

- Hybrid transformer
  - Two-to-four-wire conversion



4.8 Digital subscriber lines

- Comparison between TCM mode and EC mode
  - EC offers a much better data-transmission performance at the expense of increase complexity.
  - However, with the recent advance in VLSI, complexity is no longer a main system concern. So in North America, the EC mode has been adopted as the basis for designing the transceiver.

4.8 Digital subscriber lines

- Other impairments to DSL
  - *ISI* and *Crosstalk*
    - *The transfer function of a twisted pair line can be approximated by*

$$|H_{\text{twist pair}}(f)|^2 = \exp\left(-\alpha\sqrt{f}\right)$$

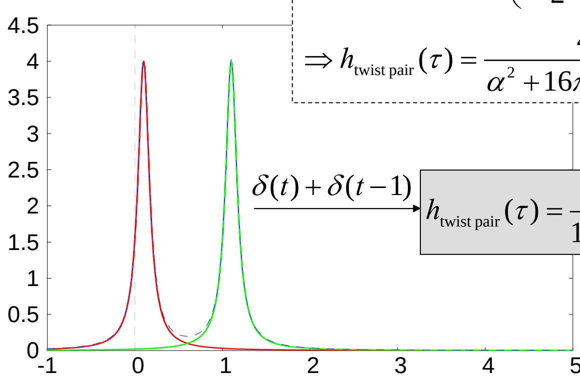
where $\alpha = k\dfrac{l}{l_0}$, k is a physical constant of the twisted pair, and l_0 and l are respectively the reference length and actual length of the twisted pair.

4.8 Digital subscriber lines

- **ISI**

$$H_{\text{twist pair}}(f) = \exp\left(-\frac{\alpha}{2}\sqrt{|f|}\right)\exp\left(-j2\pi f\tau_0\right)$$

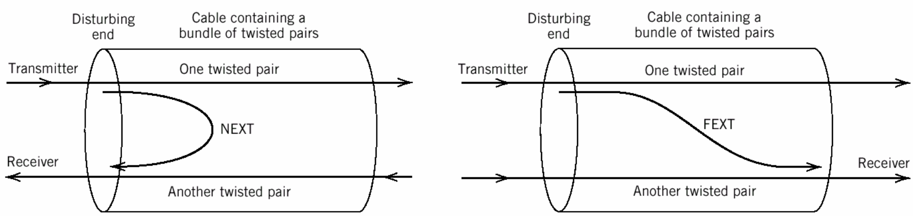
$$\Rightarrow h_{\text{twist pair}}(\tau) = \frac{4\alpha}{\alpha^2 + 16\pi^2(\tau-\tau_0)^2}$$



$$\delta(t)+\delta(t-1) \longrightarrow \boxed{h_{\text{twist pair}}(\tau) = \frac{4}{1+16\pi^2(\tau-0.1)^2}} \longrightarrow$$

4.8 Digital subscriber lines

- Crosstalk
  - Capacitive coupling that exists between adjacent twisted pairs in a cable
    - Near-end crosstalk and Far-end crosstalk

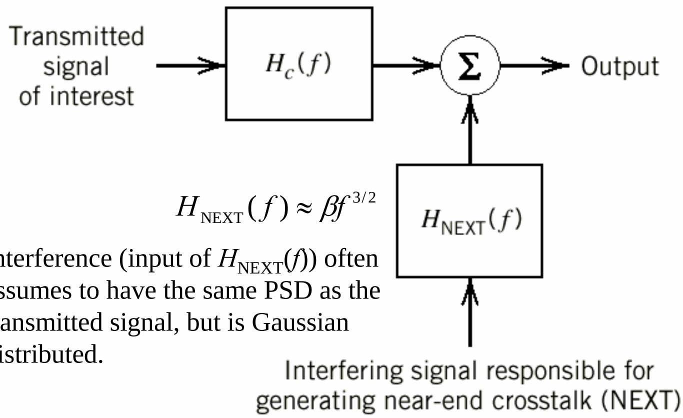


Near-end crosstalk   Far-end crosstalk

4.8 Digital subscriber lines

- Crosstalk (cont.)
  - FEXT suffers the same line loss as the signal, whereas NEXT does not.
    - This is close to the phenomenon of *near-far effect* of wireless channel.
  - Accordingly, NEXT will be a more serious problem than FEXT. So we can ignore the effect of FEXT, and add NEXT filter to the twisted pair channel model (as shown in the figure in the next slide).

4.8 Digital subscriber lines



$$H_{\text{NEXT}}(f) \approx \beta f^{3/2}$$

Interference (input of $H_{\text{NEXT}}(f)$) often assumes to have the same PSD as the transmitted signal, but is Gaussian distributed.

4.8 Digital subscriber lines

- Other features of DSL channel
  - The PSD of the transmitted signal should be zero at zero frequency because no DC transmission through a *hybrid transformer* is possible.
  - The PSD of the transmitted signal should be low at high frequencies because
    - transmission attenuation in a twisted pair is most severe at high frequency;
    - crosstalk due to capacitive coupling between adjacent twisted pairs increases dramatically at high frequency (recall that the impedance of a capacitor is inversely proportional to frequency).

4.8 Digital subscriber lines

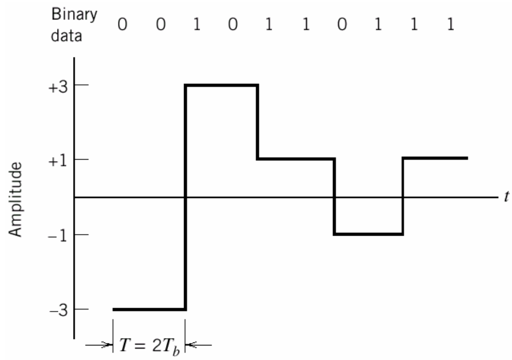
- Possible candidates for line codes that are suitable for DSL
  - Manchester code
    - Zero DC component but large spectrum at high frequency so it is vulnerable to NEXT and ISI.
  - Bipolar (Alternate mark inversion or AMI) code
    - Successive 1s are represented alternately by positive and negative but equal levels, and 0 is represented by a zero level.
    - Zero DC component. Its NEXT and ISI performance is slightly inferior to the *modified duobinary code* on all digital subscriber loops.

4.8 Digital subscriber lines

- Possible candidates for line codes that are suitable for DSL
  - Modified duobinary code
    - Of no DC component and moderately spectrally efficient. However, its robustness against NEXT and ISI is about 2 to 3 dB poorer than that of (2B1Q) block codes on worst-case subscriber lines.
  - 2B1Q code
    - Two binary bits encoded into one quaternary symbol (four-level PAM signal).
    - Zero DC component, and offers the best performance among all the codes introduced. So it is adopted as the standard as the North American standard for DSL.

4.8 Digital subscriber lines

- Possible candidates for line codes that are suitable for DSL
  - 2B1Q code (cont.)



Dibit	Amplitude
00	−3
01	−1
11	+1
10	+3

4.8 Digital subscriber lines

- 2B1Q code (cont.)
  - With 2B1Q line coding, adaptive equalizer and echo cancellation, it is possible to achieve BER = 10^{-7} operating full duplex at 160 kb/s.

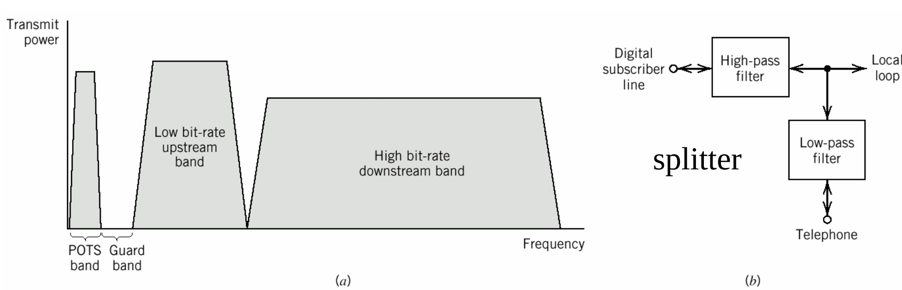
4.8 Asymmetric digital subscriber lines

- ADSL is targeted to simultaneously support three services at a single twisted-wire pair
  - Data transmission downpstream at 9 Mbps
  - Data transmission upstream at 1Mpbs
  - Plain old telephone service (POTS)

- Some notes
  - It is named *asymmetric* because the downstream bit rate is much higher than the upstream bit rate.
  - The actually achievable bit rates depend on the length of the twisted pair used to do the transmission.

4.8 Asymmetric digital subscriber lines

- Frequency-division multiplexing (FDM) technique is used to combine analog voice and DSL data.
- Upstream and downstream data transmission are placed in different frequency band to avoid crosstalk.



4.8 Asymmetric digital subscriber lines

- Various applications can be applied to asymmetric transmissions, such as video-on-demand (VoD).
  - For example
    - Downstream = 1.544 Mbps (DS1) for video data
    - Upstream = 160 kbps for real-time control commands.

4.9 Optimum linear receiver

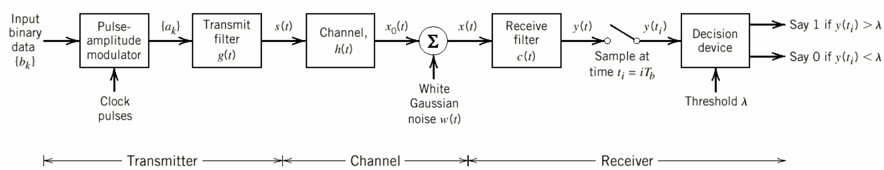
- Zero-forcing equalizer
  - One receiver design is to use a *zero-forcing equalizer* followed by a decision-making device.
  - The design objective of zero-forcing equalizer is to force the ISI to zero at all sampling instances $t = kT_b$ for $k \neq 0$, provided that "the channel noise $w(t)$ is zero."

4.9 Optimum linear receiver

- Zero-forcing equalizer (cont.)
  - This reduces to the *Nyquist criterion*.

$$\sum_{n=-\infty}^{\infty} P\left(f-\frac{n}{T_b}\right)=T_b \quad \text{or} \quad p(nT_b)=\begin{cases}1, & n=0\\ 0, & n\neq 0\end{cases}$$

where $P(f) = G(f)H(f)C(f)$.



4.9 Optimum linear receiver

- Zero-forcing equalizer (cont.)
  - A serious consequence of the ignorance of $w(t)$ in the design of zero-forcing equalizer is the performance degradation due to *noise enhancement*.

4.9 Optimum linear receiver

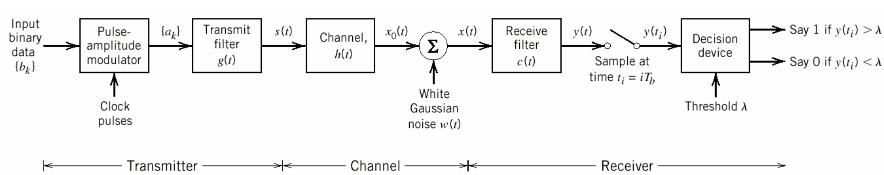
- Example of *noise enhancement*.
  - Suppose that the receiver filter is a tapped-delay-line equalizer, which is of the form

$$c(t) = \sum_{k=0}^{\infty} c_k \delta(t - kT_b)$$

  - Assume ideally that $G(f) = 1$.
  - Hence, the Nyquist criterion becomes:

$$p(nT_b) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

where $P(f) = H(f)C(f)$.

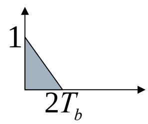


$$\begin{aligned}
p(t) &= \int_{-\infty}^{\infty} h(\tau) c(t-\tau) d\tau \\
&= \int_{-\infty}^{\infty} h(\tau) \left(\sum_{k=0}^{\infty} c_k \delta(t - \tau - kT_b) \right) d\tau \\
&= \sum_{k=0}^{\infty} c_k \int_{-\infty}^{\infty} h(\tau) \delta(t - \tau - kT_b) d\tau \\
&= \sum_{k=0}^{\infty} c_k h(t - kT_b)
\end{aligned}$$

$$p_n = p(nT_b) = \sum_{k=0}^{\infty} c_k h((n-k)T_b) = \sum_{k=0}^{\infty} c_k h_{n-k} = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

It is reasonable to assume that $h_n = 0$ for $n < 0$, and $h_0 = 1$.

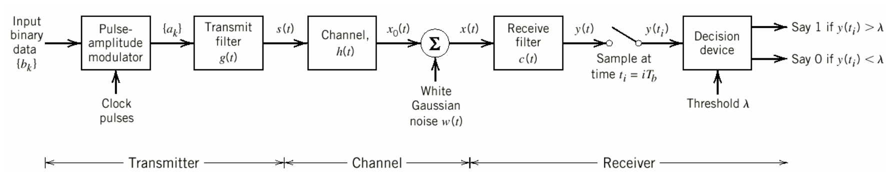
$$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ h_1 & 1 & 0 & \cdots & 0 \\ h_2 & h_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_N & h_{N-1} & h_{N-2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \\ c_N \end{bmatrix} \text{ for arbitrary } N > 0.$$

Suppose 

$$h(\tau) = \begin{cases} 1 - |\tau| / (2T_b), & 0 \le \tau < 2T_b \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow h_0 = 1, h_1 = \frac{1}{2}, \text{ and } h_n = 0 \text{ for } n \ne 0,1.$$

$$\Rightarrow c_n = (-1)^n 2^{-n} \text{ for } n \ge 0, \text{ and zero, otherwise.}$$



The above $c(t)$ can successfully remove ISI, provided $w(t) = 0$. Now add the additive white Gaussian noise $w(t)$, which also passes the filter $c(t)$.

At any time instance nT_b, the sampled noise becomes

$$\int_{-\infty}^{\infty} w(\tau) c(nT_b - \tau) d\tau = \sum_{k=0}^{\infty} c_k w_{n-k}$$

The sampled noise variance then becomes:

$$\mathrm{Var}\left[\sum_{k=0}^{\infty} c_k w_{n-k}\right] = \sum_{k=0}^{\infty} c_k^2 \mathrm{Var}[w_{n-k}] = \sigma_w^2 \sum_{k=0}^{\infty} 2^{-2k} = \frac{4}{3}\sigma_w^2 > \sigma_w^2$$

- An easier way to interpret the noise enhancement phenomenon.
  - Nyquist criterion requires that:

$$\sum_{n=-\infty}^{\infty} P\left(f-\frac{n}{T_b}\right)=\sum_{n=-\infty}^{\infty} H\left(f-\frac{n}{T_b}\right)C\left(f-\frac{n}{T_b}\right)=T_b$$

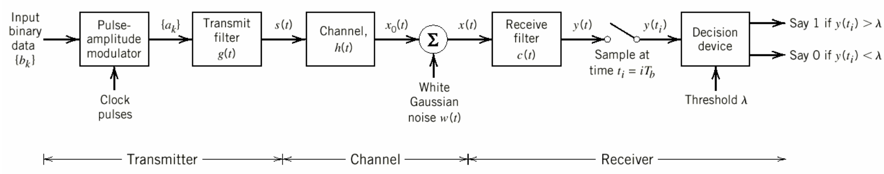
  - A sufficient condition for the Nyquist criterion is that:

$$H(f)C(f)=\text{Raised Cosine Spectrum}$$

  - When $H(f)$ is very small (or zero) at some frequency range, $C(f)$ has to be very large (or infinity) at the same frequency range in order to "equalize" the spectrum.
  - Thus, the noise spectrum $S_W(f)|C(f)|^2$ after passing through $C(f)$ will be "enhanced."

4.9 Optimum linear receiver

- To alleviate noise enhancement phenomenon, it is better to simultaneously consider the ISI and channel noise.
- An approach of this kind is to use the *mean-square error criterion*, and find a balanced solution to the problem of reducing the effects of both channel noise and intersymbol interference.



$$\begin{cases} y(t) = c(t) * x(t) = \int_{-\infty}^{\infty} c(\tau)x(t-\tau)d\tau \\ x(t) = \sum_k a_k q(t - kT_b) + w(t) \\ q(t) = g(t) * h(t) \end{cases}$$

$$\Rightarrow y(iT_b) = \sum_k a_k \int_{-\infty}^{\infty} c(\tau)q(iT_b - \tau - kT_b)d\tau + \int_{-\infty}^{\infty} c(\tau)w(iT_b - \tau)d\tau = \xi_i + n_i$$

For perfect receiver, $y(iT_b) = a_i$.

So, the error $e_i = (\xi_\text{i} + n_i) - a_i$.

The mean squared error criterion then wishes to minimize:

$$J_i = \frac{1}{2}E[e_i^2] = \frac{1}{2}E[\{(\xi_\text{i} + n_i) - a_i\}^2],$$

where the factor $\frac{1}{2}$ is added for convenience.

$$J_i = \frac{1}{2}E[\xi_i^2] + \frac{1}{2}E[n_i^2] + \frac{1}{2}E[a_i^2] + E[\xi_i n_i] - E[n_i a_i] - E[\xi_i a_i]$$

1st term

For i.i.d. $\{a_k\}$ where $a_k = \pm 1$,

$$E[\xi_i^2] = \sum_k \sum_l E[a_k a_l] \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(\tau_1)c(\tau_2)q(iT_b - kT_b - \tau_1)q(iT_b - lT_b - \tau_2)d\tau_1 d\tau_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(\tau_1)c(\tau_2)R_q(\tau_1, \tau_2; i)d\tau_1 d\tau_2$$

where $R_q(\tau_1, \tau_2; i) = \sum_k q(iT_b - kT_b - \tau_1)q(iT_b - kT_b - \tau_2)$

2nd term

Assume white $w(t)$ with PSD $N_0/2$.

$$E[n_i^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(\tau_1)c(\tau_2)E[w(iT_b-\tau_1)w(iT_b-\tau_2)]d\tau_1 d\tau_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(\tau_1)c(\tau_2)\frac{N_0}{2}\delta(\tau_1-\tau_2)d\tau_1 d\tau_2$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty} c^2(\tau_1)d\tau_1$$

3rd term

For i.i.d. $\{a_k\}$ where $a_k = \pm 1$, $E[a_i^2]=1$.

4th and 5th term

By the independence of $\{a_k\}$ and $w(t)$, and zero mean of n_i,

$$E[\xi_i n_i] = E[\xi_i]E[n_i] = 0 \text{ and } E[n_i a_i] = E[n_i]E[a_i] = 0.$$

6th term

$$E[\xi_i a_i] = \sum_k E[a_k a_i]\int_{-\infty}^{\infty} c(\tau)q(iT_b-kT_b-\tau)d\tau = \int_{-\infty}^{\infty} c(\tau)q(-\tau)d\tau$$

Substitute all six terms into J_i.

$$J_i = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(\tau_1)c(\tau_2)R_q(\tau_1,\tau_2;i)d\tau_1 d\tau_2$$

$$+\frac{N_0}{4}\int_{-\infty}^{\infty} c^2(\tau_1)d\tau_1 + \frac{1}{2} - \int_{-\infty}^{\infty} c(\tau_1)q(-\tau_1)d\tau_1$$

Taking the derivative of J_i with respect to $c(s)$, we obtain:

$$\frac{\partial J_i}{\partial c(s)} = \frac{1}{2}\int_{\tau_2\in\Re/\{s\}} c(\tau_2)R_q(s,\tau_2;i)d\tau_2 + \frac{1}{2}\int_{\tau_1\in\Re/\{s\}} c(\tau_1)R_q(\tau_1,s;i)d\tau_1$$

$$+ c(s)R_q(s,s;i) + \frac{N_0}{2}c(s) - q(-s)$$

Since $R_q(\tau_1,\tau_2;i)=\sum_k q(iT_b-kT_b-\tau_1)q(iT_b-kT_b-\tau_2)=R_q(\tau_2,\tau_1;i)$,

$$\frac{\partial J_i}{\partial c(s)}=\int_{-\infty}^{\infty}c(\tau)R_q(s,\tau;i)d\tau+\frac{N_0}{2}c(s)-q(-s)$$
$$=\int_{-\infty}^{\infty}c(\tau)\left(R_q(s,\tau;i)+\frac{N_0}{2}\delta(s-\tau)\right)d\tau-q(-s)=0$$

So the optimal design for equalizer (or received filter) $c(\tau)$ is:

$$\int_{-\infty}^{\infty}c(\tau)\left(R_q(s,\tau;i)+\frac{N_0}{2}\delta(s-\tau)\right)d\tau=q(-s)$$

where $R_q(s,\tau;i)=\sum_k q(iT_b-kT_b-s)q(iT_b-kT_b-\tau)$ and $q(t)=g(t)*h(t)$.

An equalizer designed based on the above equality is referred to as the *minimum-mean square error* (mmse) equalizer.

Observe that $R_q(\tau_1,\tau_2;i)=\sum_k q(iT_b-kT_b-\tau_1)q(iT_b-kT_b-\tau_2)$ only depends on the difference between τ_1 and τ_2, and is invariant with respect to i.

We can then reexpress it as $R_q(\tau_1-\tau_2)$.

Take this into the design criterion for mmse equalizer to yield:

$$\int_{-\infty}^{\infty}c(\tau)\left(R_q(s-\tau)+\frac{N_0}{2}\delta(s-\tau)\right)d\tau=q(-s)$$

We can then take the Fourier transform of both sides, and obtain:

$$C(f)\left(S_q(f)+\frac{N_0}{2}\right)=Q^*(f)$$

$$\Rightarrow C(f)=\frac{Q^*(f)}{S_q(f)+N_0/2}$$ solution for mmse equalizer.

4.9 MMSE equalizer

- Summary
  - The MMSE equalizer can be viewed as the concatenation of two filters:
    - a matched filter $Q^*(f)$ to $Q(f) = G(f)H(f)$
    - an equalizer whose frequency response is the inverse of $S_q(f) + N_0/2$.

4.9 MMSE equalizer

- Property of $S_q(f)$
  - The text wrote that $S_q(f) = \frac{1}{T_b}\sum_k \left| Q\left(f + \frac{k}{T_b} \right) \right|^2$, which is periodic with period $1/T_b$. This implies that $R_q(\tau)$ consists of a series of pulse train with width T_b, which is not true.

$$\boxed{R_q(\tau_1 - \tau_2) = \sum_k q(kT_b - \tau_1) q(kT_b - \tau_2)}$$

$$S_q(f) = \int_{-\infty}^{\infty} R_q(\tau)\exp(-j2\pi f\tau)d\tau = \int_{-\infty}^{\infty}\left(\sum_k q(kT_b-\tau)q(kT_b)\right)\exp(-j2\pi f\tau)d\tau$$

$$= \sum_k q(kT_b)\int_{-\infty}^{\infty} q(kT_b-\tau)\exp(-j2\pi f\tau)d\tau$$

$$= \sum_k q(kT_b)\int_{-\infty}^{\infty} q(v)\exp(-j2\pi f(kT_b-v))dv$$

$$= \sum_k q(kT_b)\exp(-j2\pi fkT_b)\int_{-\infty}^{\infty} q(v)\exp(j2\pi fv)dv$$

$$= Q^*(f)\sum_k q(kT_b)\exp(-j2\pi fkT_b)$$

$$= Q^*(f)\int_{-\infty}^{\infty}\left(\sum_k q(t)\delta(t-kT_b)\right)\exp(-j2\pi ft)dt$$

$$= Q^*(f)\cdot\frac{1}{T_b}\sum_k Q\left(f+\frac{k}{T_b}\right)$$

4.9 Implementation of MMSE equalizer

- One can approximate $1/[S_q(f) + N_0/2]$ by a periodic function with:

$$S_q(f) = Q^*(f)\cdot\frac{1}{T_b}\sum_k Q\left(f+\frac{k}{T_b}\right) \approx \frac{1}{T_b}\sum_k\left|Q\left(f+\frac{k}{T_b}\right)\right|^2 \equiv \tilde{S}_q(f)$$

- Since $\Theta_q(f) = 1/[\tilde{S}_q(f)+N_0/2]$ is now periodic with period $1/T_b$, we obtain by Fourier series that

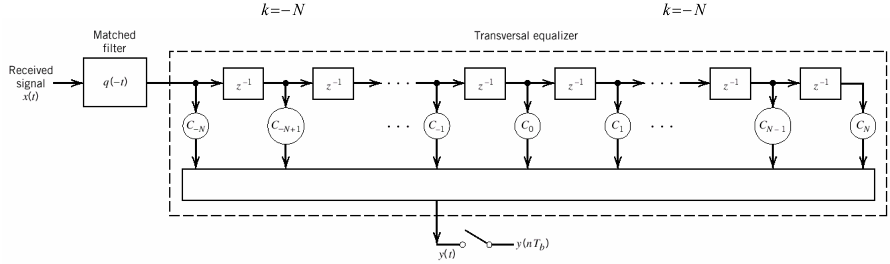
$$\Theta_q(f) = \sum_{k=-\infty}^{\infty} c_k \exp(j2\pi kfT_b)$$

where $c_k = T_b\int_{-1/(2T_b)}^{1/(2T_b)}\Theta_q(f)\exp(-j2\pi kfT_b)df$.

4.9 Implementation of MMSE equalizer

- We can approximate $Q_q(f)$ by its main $2N+1$ terms as:

$$\Theta_q(f) \approx \sum_{k=-N}^{N} c_k \exp(j2\pi k f T_b) \Rightarrow \theta_q(\tau) \approx \sum_{k=-N}^{N} c_k \delta(t + kT_b)$$



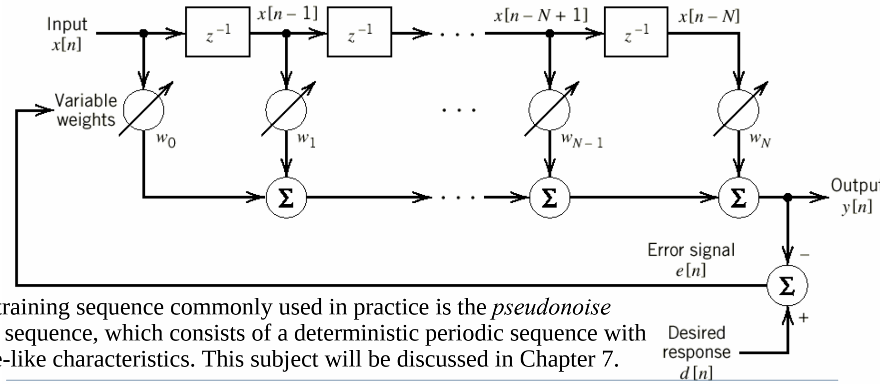
One can therefore approximate $1/[S_q(f) + N_0/2]$ by a transversal tapped-delay-line equalizer

4.9 Implementation of MMSE equalizer

- Final notes
  - In real-life telecommunications environment, the channel is usually time-varying.
  - Therefore, an *adaptive receiver* that provides for the adaptive implementation of both the matched filter and the equalizer in a combined manner is usually necessary.
  - It may be desirable to have the taps of the equalizer (named *fractionally spaced equalizer* or FSE) spaced by a fraction of the symbol period T_b, e.g., $T_b/2$.
    - $S_q(f)$ is now approximated by a periodic spectrum with period $2/T_b$. Details are omitted in this book.

4.10 Adaptive equalization

- The equalizer is adjusted under the guidance of a *training sequence* transmitted through the channel.



The training sequence commonly used in practice is the *pseudonoise* (PN) sequence, which consists of a deterministic periodic sequence with noise-like characteristics. This subject will be discussed in Chapter 7.

4.10 Adaptive equalization

- Least-mean-square (LMS) algorithm

$$e[n] = d[n] - y[n] = d[n] - \sum_{k=0}^{N} w_k x[n-k]$$

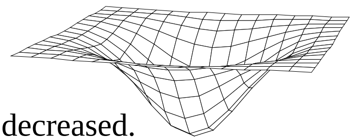
- Design objective
  - To find the filter coefficients w_0, w_1, …, w_N so as to minimize *index of performance J*:

$$J = e^2[n]$$

4.10 Adaptive equalization

- To minimize J, we should update w_i toward the bottom of the J-bowel.

$$g_i \equiv \frac{\partial J}{\partial w_i}$$



  - So when $g_i > 0$, w_i should be decreased.
  - On the contrary, w_i should be increased if $g_i < 0$.
  - Hence, we may define the update rule as:

$$\hat{w}_{i,\text{next}} = \hat{w}_{i,\text{current}} - \frac{1}{2}\mu \cdot g_i$$

where μ is a chosen constant step size, and ½ is included only for convenience of analysis.

4.10 Adaptive equalization

$$J = \left(d[n] - \sum_{k=0}^{N} w_k x[n-k] \right)^2$$

$$= d^2[n] - 2\sum_{k=0}^{N} w_k d[n]x[n-k] + \sum_{k=0}^{N}\sum_{j=0}^{N} w_k w_j x[n-k]x[n-j]$$

$$g_i = \frac{\partial J}{\partial w_i} = -2d[n]x[n-i] + 2\sum_{k=0}^{N} w_k x[n-k]x[n-i]$$

$$= -2x[n-i]\left(d[n] - \sum_{k=0}^{N} w_k x[n-k] \right)$$

$$= -2x[n-i]e[n]$$

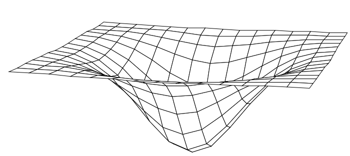
4.10 Adaptive equalization

$$\Rightarrow \text{Repeat}\begin{cases} e[n] = d[n] - \sum_{k=0}^{N} w_{k,\text{current}}\, x[n-k] \\ \text{For } 0 \le i \le N,\ w_{i,\text{next}} = w_{i,\text{current}} + \mu \cdot x[n-i]e[n]. \\ \text{For } 0 \le i \le N,\ w_{i,\text{current}} = w_{i,\text{current}} \end{cases}$$

- Some notes on LMS algorithm
  - There is no guarantee that the algorithm converges to a local minimum (could converge to a saddle point).
  - There is even no guarantee that the algorithm converges.

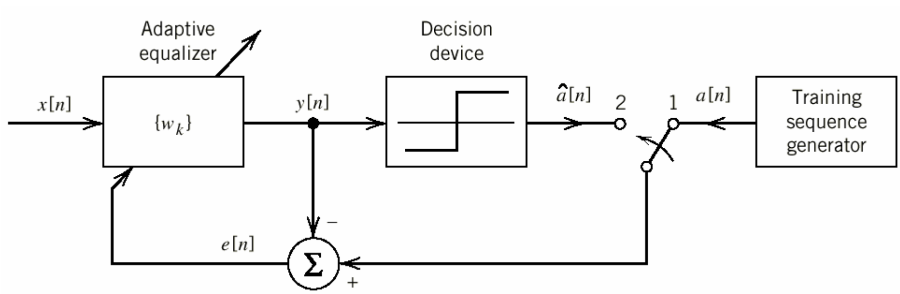
4.10 Adaptive equalization

- Some notes on LMS algorithm (cont.)
  - If μ is too large, high excess mean-square error may occur.
  - If μ is too small, a *slow rate of convergence* may arise.



4.10 Operation of the equalizer

- Two modes of operations for adaptive equalizer
  - Training mode (position 1)
  - Decision-directed mode (position 2)

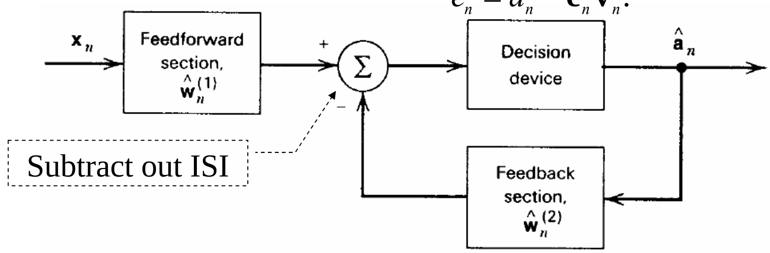


4.10 Decision-directed mode

- In normal operation, the decisions made by the receiver are correct with high probability.
- Under such premise, we can use the previous decisions to *calibrate* or *track* the tap coefficients.
- In this mode,
  - if μ is too large, high excess mean-square error may occur.
  - if μ is too small, a *too-slow tracking* may arise.
- We can further extend the idea of *decision-directed* or *decision-feedback* to the decision-feedback equalizer (DFE).

4.10 Decision-feedback equalizer

$$e_n = a_n - \mathbf{c}_n^T \mathbf{v}_n.$$



Let $\mathbf{c}_n = \begin{bmatrix} \hat{\mathbf{w}}_n^{(1)} \\ \hat{\mathbf{w}}_n^{(2)} \end{bmatrix}$ and $\mathbf{v}_n = \begin{bmatrix} \mathbf{x}_n \\ \hat{\mathbf{a}}_n \end{bmatrix}$, where $n =$ sample time at nT.

Denote $e_n = a_n - \mathbf{c}_n^T \mathbf{v}_n$, where a_n is the nth transmitted symbol.

4.10 Decision-feedback equalizer

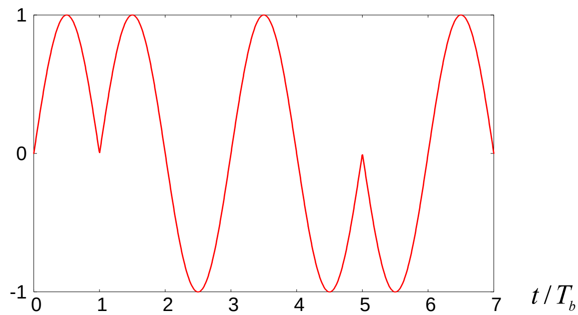
- Then DFE gives:

$$\begin{bmatrix} \hat{\mathbf{w}}_{n+1}^{(1)} \\ \hat{\mathbf{w}}_{n+1}^{(2)} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{w}}_{n}^{(1)} \\ \hat{\mathbf{w}}_{n}^{(2)} \end{bmatrix} + \begin{bmatrix} \mu_1 e_n \mathbf{x}_n \\ \mu_2 e_n \hat{\mathbf{a}}_n \end{bmatrix}.$$

- As anticipated, DFE suffers from error propagation due to incorrect decisions.
- However, error propagation will not persist indefinitely; rather, it tends to occur in *bursts*.
  - E.g., if the number of taps in the feedback section is L, then the influence of one decision error will be flushed out after subsequent L correct decisions.

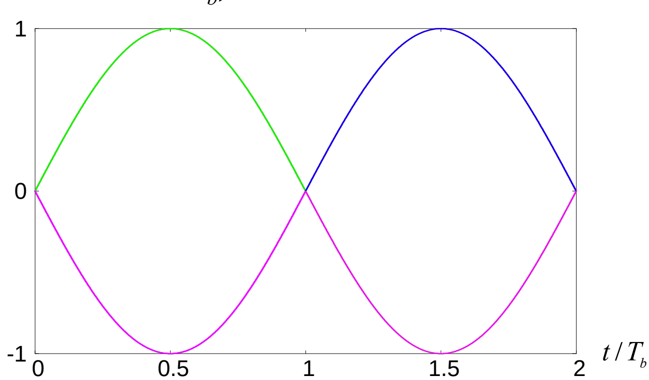
4.11 Computer experiments: Eye patterns

- Eye pattern: The synchronized superposition of all possible realizations of the signal of interest viewed within a particular signaling interval.



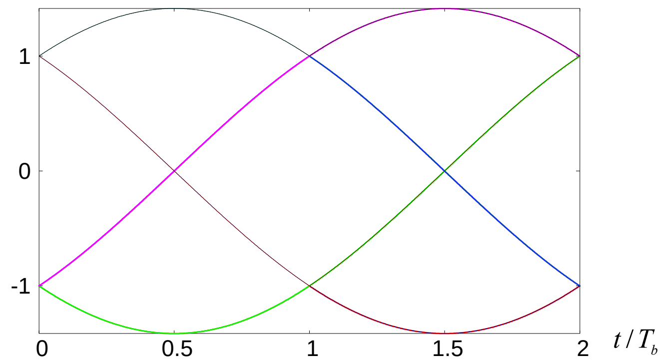
4.11 Computer experiments: Eye patterns

- Eye pattern for pulse shaping function $p(t)$ is half-cycle sine wave with duration T_b, and error-free BPSK transmission.

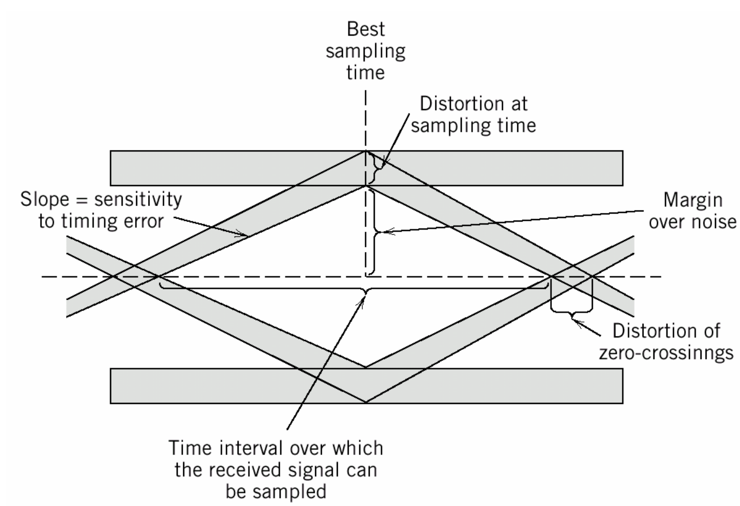


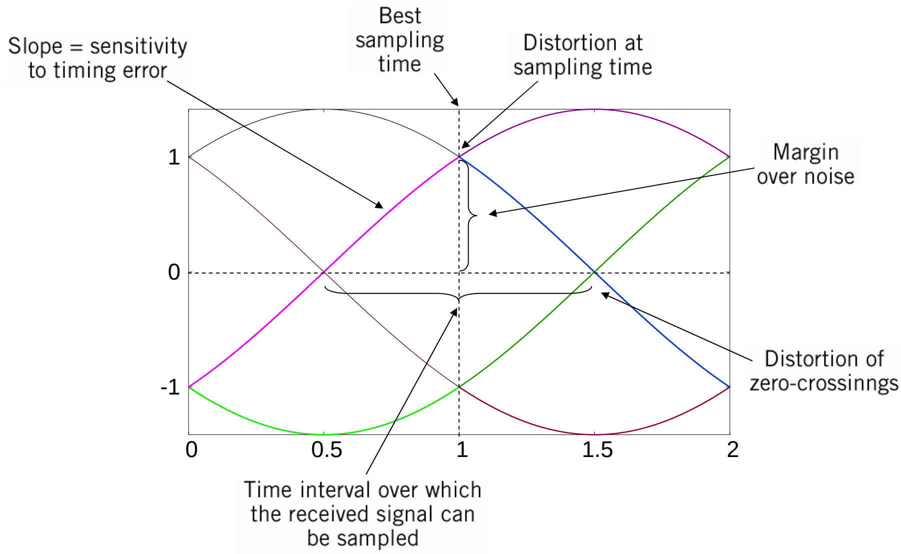
4.11 Computer experiments: Eye patterns

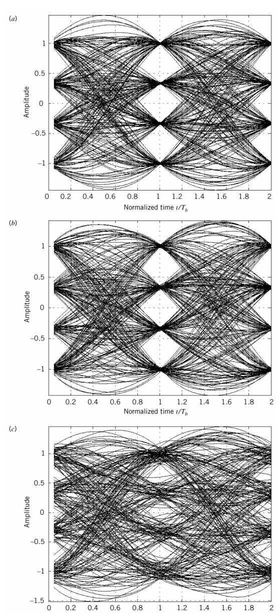
- Eye pattern for pulse shaping function $p(t)$ is half-cycle sine wave with duration $2T_b$, and error-free BPSK transmission.



Interpretation of eye pattern







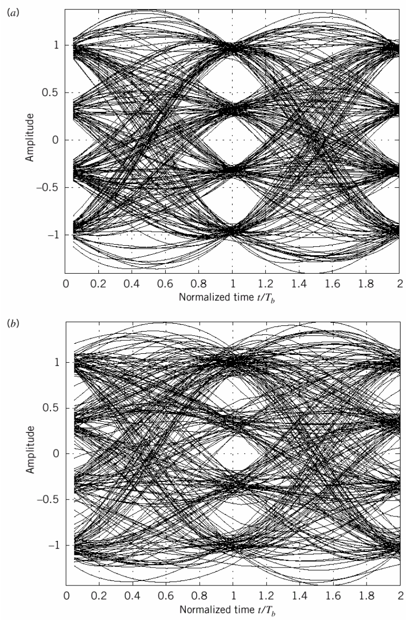
Experiment 1: Effect of channel noise

(*Raise-cosine pulse-shaping with roll-off factor* $\alpha = 0.5$, W = 0.5 Hz, M = 4)

(*a*) Eye diagram for noiseless quaternary system.

(*b*) Eye diagram for quaternary system with SNR = 20 dB.

(*c*) Eye diagram for quaternary system with SNR = 10 dB.



Experiment 2: Effect of bandwidth limitation

(*Raise-cosine pulse-shaping with roll-off factor* $\alpha = 0.5$, W = 0.5 Hz, $M = 4$)

(a) Eye diagram for noiseless band-limited quaternary system: cutoff frequency $f_o = 0.975$ Hz

(b) Eye diagram for noiseless band-limited quaternary system: cutoff frequency $f_o = 0.5$ Hz

(The channel is now modeled by a low-pass *Butterworth filter* with

$$|H(f)|^2 = \frac{1}{1+(f/f_0)^{50}}$$